Experiment VII — Forced Harmonic Motion

I. Preparing for Lab

The purpose of this lab is to examine the resonance and properties of a driven mass-on-a-spring harmonic oscillator.

To prepare for this lab before your session starts, read through the Physical Theory section below; for further reference, see sections 16.7 – 16.8 in your textbook.

*Finally, you must complete the **Pre-Lab questions** on **Expert TA** before your lab starts.*

Equipment:

Short air-track	blower and hose
GoMotion sonic ranger	cart with reflector and damping magnets
Two springs	hooks for attaching springs to cart
Ruler	banana plugs
Speaker	function generator with amplifier
LabPro with voltage pickup	Excel template for Lab 7
Logger Pro template for Forced Harmonic Motion	

II. Physical Theory

When a mass on a spring is driven in forced harmonic motion at a frequency near its natural oscillation frequency***, the amplitude will perpetually increase*** until some other physical factor limits its size or stops the motion. This phenomenon is called resonance, and for this reason the natural oscillation frequency is also known as **resonant frequency.** Resonance due to forced harmonic motion serves as a model for many important systems in physics and engineering. Examples range from resonant electrical circuits used in tuners, to the oscillations of buildings and bridges during earthquakes, to resonant optical cavity use in lasers. Although these phenomena occur in different types of systems, they all adhere to the same equations.

Any object disturbed from equilibrium and then subject to a restorative force and will undergo harmonic oscillation. In the case of a mass attached to a spring, the restorative force is the resistive elastic force of the spring on the mass. At equilibrium the spring is neither stretched nor compressed and so does not exert a force on the mass. In order to keep the calculations simple, we will let this neutral spring position be $x = 0$. In that case the spring force is

$$F_s = -kx \,, \tag{1}$$

where the negative sign is due to the resistive nature of the spring: when the object is out of equilibrium, the spring wants to push or pull the mass in the opposite direction back toward equilibrium, i.e. when the cart is to the right of $x = 0$, the force is to the left, and vice versa.

Any realistic oscillator also experience air drag and other forms of friction, leading to a decay of the oscillation amplitude with time; such motion is called **damped oscillation**. Assuming that the drag force F_d is proportional to the velocity of the mass, we can write that force as

$$F_d = -b\frac{dx}{dt} \,, \tag{2}$$

where b is a positive constant and the negative sign means that the drag is opposite to the direction the cart is moving. According to Newton's second law and equations (1)-(2), the acceleration of the cart will be

$$\frac{d^2x}{dt^2} = -\frac{b}{m}\frac{dx}{dt} - \frac{k}{m} \,. \tag{3}$$

Equation (3) is a linear homogenous second-order differential equation. The solutions to this equation are well-known

$$x = x_0 \sin(2\pi f_{osc} t - \varphi) e^{-t/\tau_d} \,, \tag{4}$$

where the **damping time** $\boldsymbol{\tau_d}$ is

$$\tau_d = \frac{2m}{b} \,, \tag{5}$$

and the **natural oscillation frequency** f_{osc} is

$$f_{osc} = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \,. \tag{6}$$

The **phase** ϕ and the **amplitude** x_0 are set by the position and velocity of the mass at $t = 0$, which act as boundary conditions on the solution of Equation (3).

We can also write the oscillation frequency as

$$f_{osc} = f_0\sqrt{1 - \frac{1}{4Q^2}} \tag{7}$$

where the f_0 is the oscillation frequency **without damping**

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{8}$$

and the **quality factor** Q is:

$$Q = 2\pi f_0 \frac{m}{b} = 2\pi f_0 \frac{\tau_d}{2} \tag{9}$$

The quality factor Q is a pure number that characterizes energy loss in the system. For Q >> 1 the system has low loss and will undergo many oscillations (of order Q) before its energy is damped away.

It is also useful to define the **period** T_{osc} of the natural oscillation:

$$T_{osc} = \frac{1}{f_{osc}} = \frac{2\pi}{\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}} \quad . \qquad (10)$$

Equation (10) implies that as the mass *m* increases or the spring constant *k* gets weaker, the period increases.

Driven Motion

To maintain oscillation over long periods of time despite damping, one needs to drive the oscillations by some external means. For this to work, the driver will need to operate a frequency similar to the natural oscillation frequency. For a driver with amplitude A_0 and **drive frequency** f, then Eqn (3) becomes

$$\frac{d^2x}{dt^2} = -\frac{b}{m}\frac{dx}{dt} - \frac{k}{m}x + \frac{k}{m}A_0 \sin(2\pi f t) \qquad (11)$$

If the driver starts at $t = 0$, there will be some **transient motion** initially that will damp out on the time scale τ_d; for times much longer than τ_d, the system will settle into a regular oscillatory motion at the same frequency as the drive and the solution to Equation (11) becomes

$$x(t) = x_0 \sin(2\pi ft - \theta), \qquad (12)$$

where the **amplitude** $\boldsymbol{x_0}$ of the mass oscillation is related to the drive frequency f and drive amplitude A_0 by

$$x_0 = \frac{A_0/2}{\sqrt{\frac{f^2}{Q^2 f_0^2} + \left[\frac{f^2}{f_0^2} - 1\right]^2}} . \qquad (13)$$

and the **phase difference** θ depends on the drive frequency as

$$\theta = \arctan\left\{\frac{1}{Q}\left[\frac{f\ f_0}{f_0^2 - f^2}\right]\right\} \quad . \qquad (14)$$

This angle θ is the difference in phase between the motion of the driver and the mass. The arctan function is multi-valued, but it is conventional to choose $0 < \theta < \pi$ radians, or $0 < \theta < 180^o$.

Figure 1(a) shows a plot of Equation (13) and so the x_0 dependence on drive frequency f. The amplitude x_0 of the mass oscillation has a maximum when the drive frequency f is at the **resonant frequency** of the mass-spring system:

$$f_{res} = f_0\sqrt{1 - \frac{1}{2Q^2}} \qquad (15)$$

Notice that this is similar to Equation (7) for the natural oscillation frequency but different by the factor of 2 in the denominator under the radical, so that f_{res} and f_{osc} are not exactly equal to each other. In this experiment, however, Q is relatively large and typically you will not be able to measure the difference between f_{osc}, f_0 and f_{res}.

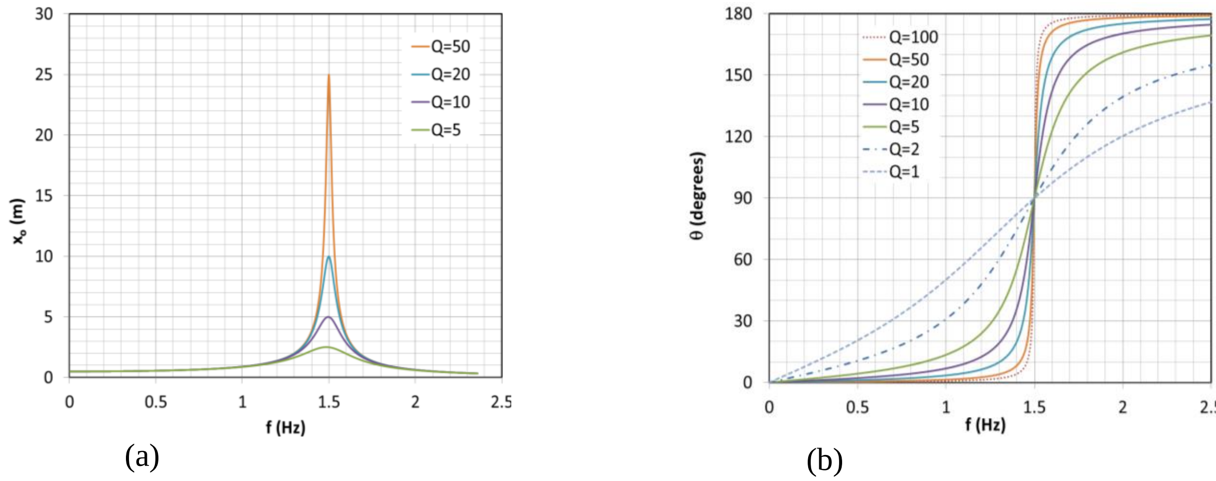

Figure 1. (a) Amplitude x_0 of harmonic oscillation of the mass as a function of frequency *f*. Here, the resonant frequency is f_o = 1.5 Hz, and curves are drawn for *Q* = 50 (highest curve), 20, 10 and 5 (lowest curve). (b) Phase θ as a function of frequency *f* for a range of *Q* values.

If a system with a large *Q* is driven at $f = f_{res}$, the amplitude of the oscillation of the system x_0 will become much larger than the driving amplitude A_0. Specifically, at resonance x_0 reaches a maximum $x_{max} \sim A_o Q$. The **full width** of the resonance peak, defined as the frequency difference between the *f*s where $x_0 = x_{max} / \sqrt{2}$, is determined by $f_{FW} = f_0/Q$.

Figure 1(b) shows plots corresponding to equation (14) showing the phase shift dependence on frequency. At *f* = 0, the phase shift is zero (**in-phase**); at resonance, the phase shift is **lagging** by 90^o; and for $f >> f_{res}$, the phase is 180^o (**out-of-phase**). Notice also that the phase changes most rapidly when the driving frequency *f* is near the resonant frequency (1.5 Hz in this case), and that the change is more rapid for larger Q.

III. Experiment

Part A: Getting started

(1) Open the Excel spreadsheet template for **Lab 7** found the Lab Templates folder on your lab station computer.

(2) Fill in your name and your lab partner's name, and choose your lab section number. *Failure to provide the correct section may result in grading problems with your report; please ask your TA if you aren't sure of your section number.*

(3) **Take a picture** of you and your partner with the lab apparatus by pointing the provide web cam appropriately and clicking "Capture image" in your lab template. If an error occurs, please consult your TA for assistance.

(4) Make sure that the indicator light on the sonic ranger is glowing green. If it isn't, check that the cord for the ranger is plugged into the LabPro interface. If not, contact your TA.

(5) Start the LoggerPro software by opening the LoggerPro Templates folder on the desktop and clicking on the file labelled **Forced Harmonic Motion**; if LoggerPro was left running from the previous lab section, close and restart it.

(6) Verify that black and red voltage leads are connected from the speaker to the CH 1 port on the side of the LabPro interface box. This connection will monitor the voltage driving the speaker.

(7) Verify that the cart has a reflector and that the two springs are connected to the cart, speaker and air-track, as seen in Figure 2.

(8) Check that the sonic ranger is pointing at the reflector on the cart and that the reflector is perpendicular to the track.

Figure 2 shows a photograph of the setup you will use. A cart of mass m is connected **between two springs** and can move left or right along an air track. The left spring goes to a **speaker**-like device that will create and drive the oscillation, and the right spring goes to a fixed point. The speaker is driven with a sine wave of frequency f , while the right spring keeps the cart centered on the track and prevents the left spring from collapsing.

A sonic ranger is used to measure the position x of the cart as a function of the time t, and we keep track of the motion of the speaker by measuring the **voltage** from the function generator creating the oscillation.

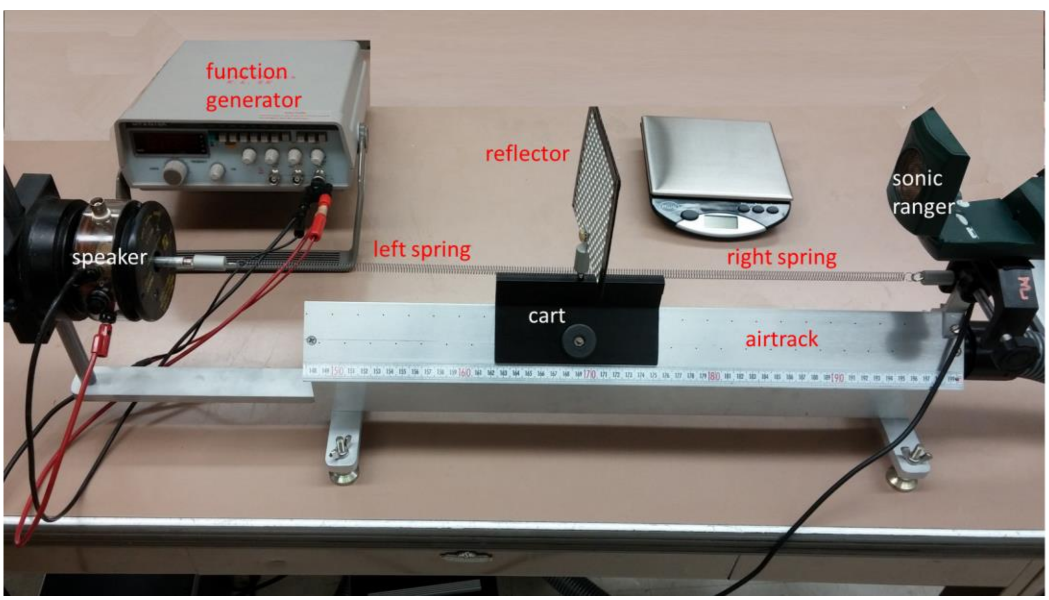

Figure 2. The apparatus for the Forced Harmonic Motion experiment.

Part B: Measuring free oscillations of the system

The goal of this part is to measure the natural oscillation of the system (when it is not being driven) and then use a macro to extract the resonant frequency and damping time from this data. You need to do this before you can do the next part of the experiment.

(1) Turn on the air blower and set the air flow control knob to about the 1 o'clock position, then push the cart a few cm and let it oscillate to verify the cart is moving with very little drag. Increasing the air flow somewhat until these oscillations last at least 10s. There should also be magnets in place on the sides of the cart to produce a small damping force and stabilize the oscillation.

(2) In the LoggerPro Forced Harmonic Motion template, you should see three columns (time, potential and position), a plot of the potential versus time, and a plot of the position versus time. Click on "Experiments", then "Data Collection", and verify that the sampling rate is set to 30 samples/second and the data collection time is set to 50s.

(3) Go back to the main panel in Logger Pro and check that the *y*-axis scale on the position vs time plot displays 0.03m to -0.03m.

(4) With the cart at rest (make sure it is at rest but do not hold it in place), **zero** the sonic ranger and the voltage probes by hitting the "0" button (next to the "Collect" button).

(5) Pull the cart about 3cm to the left on the track, let it go; check that the oscillations are relatively stable, then hit the "Collect button". LoggerPro will acquire data for x vs t. You should see a smooth decaying oscillation in the plot. When the data collection finishes, copy only the position and time data and paste them into the designated columns in Part B of your spreadsheet.

(6) Plot x vs t and label the axes. Be sure to use the standard scatter plot, but this time set the data to a smooth curve without points; this will be helpful for fitting later.

(7) From your plot, estimate the initial amplitude x_0, the oscillation frequency f_{osc}, and the decay time τ_d; if you zeroed the ranger correctly in step 4, the initial amplitude should be the maximum x value on the plot. The oscillation frequency can be estimated by finding the period (time between peaks) and taking the inverse. The decay time is the amount of time that it takes for the initial amplitude to decrease to about 1/3 of its original value (it is probably in the range 10 - 30s). Enter these values into the upper table in part B. Leave the phase ϕ, C, and χ^2 blank for now.

(8) Add the theory equation

$$x_{th} = x_0 \sin(2\pi f_{osc} t + \varphi) e^{-t/\tau_d} + C$$

to the top cell in the designated column, referring to the cells of the fitting parameters in the top table. Add x_{th} vs t to your plot and reformat the points to be a smooth curve without markers.

(9) Adjust the values of the 5 fit parameters (x_0, f_{osc}, τ_d, ϕ, and C) until the theory somewhat matches the data, then click on the macro button to do a least squares fit to your data; this will fine-tune your estimates for the fit parameters and provide you with a list of frequencies to use in the following part of the experiment.

If the fit doesn't look OK, check your estimates of the fitting parameters, adjust anything that looks off and then click the button again. You need to have a good data set and reasonably close starting parameter estimates for the macro to work optimally.

Part C: Measuring resonant motion of the driven system

The goal *of this part is to measure the motion of the system when it is driven at different frequencies, focusing on the neighborhood of the resonant frequency.*

(1) Verify that the low impedance output from the function generator is plugged into the speaker (see Figure 3) and that another set of black and red voltage probe leads are connected from the function generator to the LabPro interface box.

(2) Turn on the function generator, push the amplitude knob in and set the amplitude to its maximum setting; this should produce about a 1V driving amplitude. **Do not change the amplitude knob after this step.**

Push the "1" frequency range button (see Figure 3). Finally, push in the sine waveform button if it is not already pushed in.

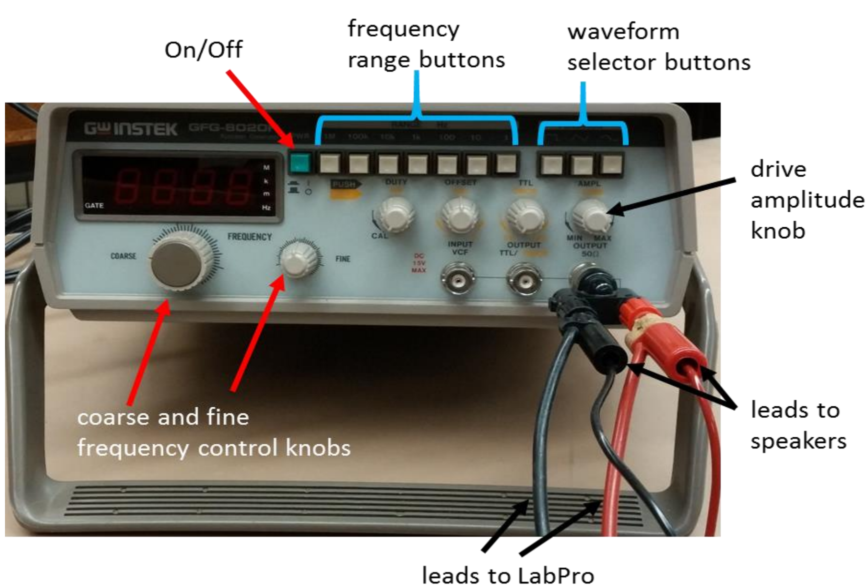

Figure 3. Function generator showing switches, knobs and leads hooked up to the low-impedance output connectors. These leads go to the drive speaker.

IMPORTANT: Do NOT select any frequency range buttons other than "1", as high frequency oscillation can damage the speaker.

(3) Turn the coarse frequency adjustment knob to set the driving frequency to about f = 1.2 Hz or 1200 mHz (note there is a bit of a delay when you turn the knob), then use the fine frequency knob (see Figure 3) to get as close to 1.2 Hz as possible.

(4) Verify that the speaker is moving back and forth by about 1-2 mm and the cart is also moving as the speaker moves (about half as much as the speaker). If the speaker is not moving, check that

- the frequency is set to about 1.2 Hz or 1200 mHz
- the amplitude knob is set to max and pushed in
- the wires from the low-impedance output of the function generator are plugged into the speaker.

If the speaker is moving but the cart is not moving, try increasing the air flow a little.

(5) Hit the "Collect" button in LoggerPro to acquire 50s worth of x vs t and V vs t. You should see nice steady oscillations in x and V. When the data collection finishes, copy all of the data x and V versus t data and paste it into the designated areas in Part C of your spreadsheet.

(6) Click the "Fit to data" macro button associated with the frequency 1.2Hz to fit the theory to your data and display the fitting parameters and the phase difference between the two waves.

This should also show you a plot of the last part of your data with the fit (this data is far downward in the spreadsheet). You can expand the plot size horizontally to see the fit better if needed. If the fits look incorrect, try retaking your data and rerunning the macro (the macro is usually pretty good at fitting to your data, but every so often it will create an erroneous fit to good data).

(7) Turn the frequency adjustment knob to set the drive frequency to the frequency indicated in the next section to the right in your template. You will need to wait a minute or so for the transient motion to decay…it typically takes this long for the oscillation to fully build up, and during this time you may see beating and other transient effects, depending on how close you are to the resonant frequency.

Once it is apparent that the motion is stabilized, repeat steps (5)-(6) for this frequency.

(8) Repeat step (7) for the rest of the driving frequencies listed in your spreadsheet. You should have collected data and fits for a total of 9 frequencies when you finish.

IV. Analysis

(1) Check that you have collected x and V versus t for all of the frequencies listed in part C of your spreadsheet.

(2) Click on the **Phase and Amplitude** button in the Analysis section of your spreadsheet template. This will run a macro that automatically extracts the amplitude x_0, phase difference θ, and driving frequency *f* for each of the data sets you created in Part C.

The output of this macro is a small table in the Analysis section at far right that summarizes the frequency, amplitude and phase difference found for the data you obtained in Part C.

(3) Make a plot of the amplitude x_0 versus frequency *f*, label the axes, and add a title.

(4) Set trial values for Q ~30-50, the drive amplitude A_0 ~ 3mm, and resonant frequency f_{res} ~ 1.3Hz in the designated table. Use these values to calculate the theory equation

$$x_{0theory} = \frac{A_0/2}{\sqrt{\frac{f_{th}^{\,2}}{Q^2 f_0^2} + \left[\frac{f_{th}^{\,2}}{f_0^2} - 1\right]^2}}$$

in the designated column. Note the factor of 1/2 difference in the numerator from eqn (13) occurs here because the mass sits **halfway between two springs** rather than at the end of a single spring.

(5) Add $x_{0theory}$ vs f_{th} to your plot and reformat the markers to be a smooth curve. Adjust the fitting parameters by hand until you get good apparent agreement between data and theory. **Note that the Q value may be much higher**… it is easiest to check the validity of your fit value for Q by *looking at the "shoulders" on either side of the peak where the curvature is highest*, rather than at the peak or the tails. If the peak gets extremely tall due to this, adjust the amplitude A_0 down to mitigate the discrepancy.

Final Question 1: *How well do your measured results for the amplitude x_0 versus the driving frequency f visually agree with theory? Did you need to make extreme adjustments to any of the fit parameters to accomplish a decent fit?*

(6) Make a plot of the phase difference θ versus frequency *f*, label the axes, and add a title.

(7) Click "Calculate θ_{th} vs f_{th}" to evaluate Equation (14) (except we added an offset θ_0 that you will need to adjust). Add the θ_{th} vs f_{th} dataset to your θ vs f plot and reformat the markers to be a smooth curve.

For this theory, use the values for f_{res}, Q and A_0 that you found in the previous step. One should usually be able to get a good fit with $\theta_0 = 0$, but you may find you need to adjust the θ_0 value to get good agreement between data and theory.

Final Question 2: *How well do your measured results for the phase θ versus the driving frequency f visually agree with theory? Did you need to use a significant value for θ_0 to accomplish a decent fit?*

Final Question 3: *How well does the natural oscillation frequency f_{osc} found by the macro in part B agree with your fit value for the resonance frequency f_{res} found in part C? Should they be exactly the same?*

Final Question 4: *Given the values for Q and f_{res} that you found in part C, use Equation (9) to calculate the damping time τ_d. Does this value agree with the value for τ_d given by the macro in part B?*

V. Finishing Up Before Leaving the Lab

(1) Record your answers to the Final Questions in your Lab spreadsheet. Yes or no questions should be justified or explained adequately.

(2) Check over your spreadsheet to make sure that you have completed everything, and that you have not missed any steps or left red feedback messages unaddressed. The automatic feedback system on the template has limited ability to detect problems, so check carefully, and consult the TA if you think your work is incorrect.

(3) Save your spreadsheet using the provided button and submit your spreadsheet on ELMS before you leave. Both partners should do this.

(4) Log out of ELMS when you are done, but **do NOT log out on the computer**, just leave it at the desktop.

Each student needs to submit a copy of their spreadsheet to their own account on ELMS before leaving the lab.