Experiment 2

Measurement and Uncertainty

I. Purpose

The purpose of this lab is to teach you how to estimate uncertainty in measurements, to learn what error propagation is, and to understand the importance of keeping track of uncertainty in measurements. You will perform a simple exercise that involves making repeated measurements of your reaction time. You will then analyze the resulting data and find a best estimate for your reaction time and the uncertainty in your measured values.

II. Preparing for the Lab

You need to prepare before going to the lab and trying to do this experiment. Start by reading through this lab write-up before you get to the lab. If you take a quick look you can see that the Introduction section for this lab is pretty long and there is not going to be enough time to read through it during the lab period - you need to read it before you get to the lab. Finally, don't forget to do the Pre-Lab questions and turn them in to Expert TA before your lab starts.

III. Pre-lab Questions *(You must submit answers to Expert TA before your section meets)*

Questions and multiple choice answers on Expert TA may vary from those given below. Be sure to read questions and choices carefully before submitting your answers on Expert TA.

#1. You measure the length of the same side of a block five times and find: b=11.0, 11.1, 11.2, 11.2 and 11.3 mm. What is the best estimate for b? Give answer to four significant figures.

#2. You measure the length of the same side of a block five times and each measurement has an uncertainty of $\delta b = 0.1$ mm. What is the uncertainty in the best estimate for b? Give answer

#3. You use a ruler marked with 1 mm increments to measure the lengths of the height h of a block and find h = 121 mm. According to the half least count rule, what is the uncertainty in your measurement of the height?

#4. A student is using a ruler to measure the length of a block that is 0.10 m long and accidentally records this as 0.010m. What type of error did the student make?

IV. Equipment

Ruler Stop-watch Excel

V. Introduction

The role of Measurements in Science

When scientists refer to "the physics" or "the chemistry" or "the biology" of some situation, they are usually referring to the underlying physical concepts and relationships that describe the phenomenon. Physicists in particular will often use mathematical equations to describe the behavior of physical quantities such as the position, velocity, momentum, or energy of an object. In this sense, "the physics" of a phenomenon may be reduced to a mathematical model. It is a remarkable fact that symbols such as x, y, z, m, v and t can be used to represent physical quantities and even more remarkable that they can be used in abstract mathematical equations to predict the behavior of objects in the real world. Putting the deep philosophical issues to the side, the key point here is that it is only by measuring physical quantities in the real world that you can connect an object's observed behavior to an abstract mathematical equation.

In fact, experimental measurements are at the very core of science: it is only through making measurements in the real world that you can decide whether a theory is or is not correct. In a traditional lab course such as this one you will sometimes be asked to determine whether a particular law of physics is actually obeyed. One soon finds that this is not so easy to do. One of the main difficulties is that a measurement of any physical quantity, such as position or velocity, can only be made to within a limited precision. That is, your measurement will provide a numerical result with a finite number of digits and the remaining digits will be unknown. For example, you might measure the length of something to be 1.5045 m. This is quite precise but there are an infinite number of digits that could follow that last 5 and they are all unknown. We say that there is some **experimental uncertainty** and define the **uncertainty** of a measurement to quantify how precisely the measured value has been determined. Since every measurement has uncertainty, it turns out that one cannot prove with mathematical rigor that a physical law is true by doing experiments. The best you can do is show that the measurements agree with theory to within the experimental uncertainty. Thus, it is essential to understand measurement errors before you can decide whether your results agree with theory.

Types of Experimental Errors

When you measure something in the lab, there are several different types of errors you can make.

Random errors involve errors in measurement due to random changes or fluctuations in the process being measured or in the measuring instrument. Random measuring errors are very common. For example, suppose you measure the length of an object using a ruler and cannot decide whether the length is closer to 10 or 11 mm. If you simply cannot tell which it is closer to, then you will tend to make a random error of about ±0.5 mm in your choice. Another example of a random error is when you try to read a meter on which the reading is fluctuating. We say that "noise" causes the reading to change with time, and this leads to a random error in determining the true reading of the meter. Random errors are the kind you will encounter most often in this course and are consequently the kind you should quote when you are asked for the uncertainty in one a measurement you make. Truly random errors are described by a ***Gaussian*** or **normal distribution**. The mathematics of this bell-shaped distribution is well beyond the level of this course but is covered in many courses on **statistics**.

A **sampling error** is a special kind of random error that occurs when you make a finite number of measurements of something which can take on a range of values. For example, the students in the university have a range of ages. You could find the exact average age of a student in the University by averaging together the ages of all of the students. Suppose instead that you only took a small sub-group, or random sample of students, and averaged together their ages. This sample average would not in general be equal to the exact average age of all of the students in the University. It would tend to be more or less close to the exact average depending on how large or small the group was. The difference between the sample average and the exact average is an example of a sampling error. In general, the larger the sample, the closer the sample average will approach the true average.

A **systematic error** is a repeated and consistent error that occurs in all of your measurements due to the design of the apparatus. Examples of systematic errors include: measuring length with a ruler which is too short, measuring time with a stopwatch which runs too fast, or measuring voltage with a voltmeter which is not properly calibrated. Systematic errors can be very difficult to detect because your results will tend to be consistent, repeatable and precise. The best way to find a systematic error is to compare your results with results from a completely different apparatus.

Illegitimate errors involve making gross mistakes in the experimental setup, in taking or recording data, or in calculating results. If you make an illegitimate mistake in these labs, you are going to lose points. Examples of illegitimate errors include: measuring time t when you were supposed to be measuring temperature T, misreading a measurement on a scale so that you think it is 2.0 when it should be 12.0, typing 2.2 into your spreadsheet when you meant to type 20.2, or using the formula "momentum = mv^2" rather than "momentum = mv". Note that the use of the word “illegitimate” here implies that a real mistake was made. On the other hand when just the word “error” is used, this does not generally imply a mistake was made. Thus the statistical meaning of the word error is not the same as the common meaning. To avoid this potential confusion it is preferable to use the words **experimental uncertainty** instead of **experimental error**, although both are commonly used by experimentalists.

How to Estimate the Uncertainty in a Single Repeatable Measurement

When taking measurements of properties such as mass or distance, using digital or analog scales on a stationary object, it is often the case that repeated measurements will yield exactly the same value over and over. In such cases it makes sense to make one measurement and then use one of the following rough rules of thumb to estimate the uncertainty in that single value.

If the scale is **analog**, such as on a ruler or a meter with a needle, one should apply the “***half least count rule***”, which says the uncertainty should be half the smallest increment marked on the scale. For example, on a typical ruler or meter stick, millimeters are the smallest increment marked; as a result, the uncertainty in a position measurement made with such a ruler will be ±0.5 mm by the rule. The convention in this manual will be to use the symbol “δ” in front of the symbol for the variable in question to name the uncertainty. For example, we can write: the uncertainty in x is $\delta x = \pm 0.5$ mm.

If the scale is **digital**, such as on an electronic balance or caliper, for simplicity we recommend for these labs that you also use the “***half least count rule***”. In this case the rule says that the uncertainty should be equal to one-half the smallest increment in the reading that the scale can make. In many situations this means the uncertainty is one-half of the smallest decimal place on the display. For example, if a scale can read the mass in grams to two decimal places, then the smallest increment will be 0.01 g and the half least count rule gives the uncertainty as $\delta g = \pm 0.005$ g. In fact, a more careful analysis of the statistics shows that for a well-built scale with a minimum increment of 0.01 g, the uncertainty can be as low as $\delta g = \pm 0.0028$ g, so the half least count rule of $\delta g = \pm 0.005$ g can be a considerable over-estimate. Nevertheless the half least count rule is simpler to use and sufficient for our purposes in these labs so we recommend using that.

Finding the Best Estimate for a Quantity by Making Repeated Measurements

Suppose you are repeatedly measuring the position x of a stationary object. Although the object is not moving, you might still find that it is not so easy to measure the position and as a result x is not exactly the same in each measurement but seems to be randomly changing from one measurement to the next. Let’s suppose you make N measurements of the quantity x and denote the result of the first measurement by x_1, the second measurement by x_2,... and the N-th measurement by x_N. If all of these measurements have the same experimental uncertainty, then the **mean** value is the **best estimate of the true value** of x. The mean value of x is just:

$$x_{avg} = \frac{1}{N}(x_1 + x_2 + ... + x_N) = \frac{1}{N}\sum_{i=1}^{N} x_i \qquad (1)$$

Notice that x_{avg} is just the **average** of a set of numbers. In EXCEL you can use Equation (1), which is pretty messy, or you can use the function **Average** to calculate the mean or average of a set of data. For example, suppose you wanted to find the average of some data that was in cells D13 to D27. You would enter the command =Average(D13:D27) in the cell where you want the average value to appear. This is easier than using Equation (1) because EXCEL figures out how many points N there are and you don't have to keep track.

__Finding the Uncertainty by Making Repeated Measurements__

Typically you should expect that a given measured value of x might be different from the true value of x because there is uncertainty in the measurement. If we denote the uncertainty by δx, then you should expect a measured value of x to typically be in a range that is about ± δx from the true value. If you have made a set of measurements of the same quantity that only differ because of random measurement uncertainty, you can use the **standard deviation** to find the uncertainty in each measurement. The standard deviation is a function that tells you how far a typical data point is from the average. If your data has a lot of scatter in it, then you will find a large standard deviation. If all of your measurements are identical, then the standard deviation will be zero. The standard deviation of set measurements of x is denoted by σ_x where:

$$\sigma = \sqrt{\frac{1}{N-1}\left((x_{avg} - x_1)^2 + (x_{avg} - x_2)^2 + ... + (x_{avg} - x_N)^2\right)} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_{avg} - x_i)^2} \quad . \qquad (2)$$

In EXCEL you can use Equation (2), which is pretty messy. or you can use the function **STDEV** to automatically calculate the standard deviation of a set of data. For example, suppose you wanted to find the standard deviation of some data which was in cells D13 to D27. You would enter the command =STDEV(D13..D27) in the cell where you want the standard deviation to appear.

There are some things about σ_x which can be confusing. For example, __why is there a factor of (N-1) in the denominator instead of N?__ To understand why there is an N-1, suppose you made just one measurement; call it x_1. In this case the average is simply $\langle x \rangle = x_1$. Notice however that Equation (2) says that the standard deviation is undefined because N=1. The reason it is undefined is because with only one measurement, it is not possible to say how much spread there is in the data. You need at least two measurements to find the standard deviation because the standard deviation needs to find out how much your data is varying!

Another thing that you might be wondering about Equation (2) is __why do we take the square each of the terms?__ If we did not take the square, but just added together all of the terms x_i - x_{avg}, we would get zero. To see this, just look at the definition of x_{avg}. The point is that data which falls below the average is balanced by data which falls above that average. Roughly speaking, by taking the square, we make all the terms positive and end up finding the (root mean square) distance of a typical data point from the average, independent of whether it is above or below the average.

Another potentially confusing thing about Equation (2): __the standard deviation σ_x does not get bigger if you measure more data points__. Recall that σ_x is just the typical distance a data point is from the average. If you take more data points, you tend to get a more accurate value for σ_x, not a bigger value. This is because that standard deviation involves how far each value is from the average value. In fact, the standard deviation reveals the **uncertainty in each measurement**. In other words, if you want to find out how uncertain a measurement is, you just need to take several measurements and then find the standard deviation. Of course this only works if the measurements vary somewhat. If they are all identical, you need to go back and use the half least count rule of thumb that is discussed above.

The Uncertainty in the Mean

When you average together N measurements of x, the average value x_{avg} will tend to be closer to the true value of x because measurements that are a bit too large will tend to be compensated for by measurements that are a bit too small. Since each measurement has a random error however, this compensating process is not exact. This means that the average value of x will be close to the true value of x, but will differ somewhat. In other words there will be an uncertainty in the average value. Assuming that each of the measurements x_i are independent and that the uncertainty in each measurement is δx, one finds

$$\delta x_{avg} = \frac{\sigma_x}{\sqrt{N}}. \qquad [3]$$

Notice that the uncertainty δx_{avg} in the mean value x_{avg} is smaller than the uncertainty σ_x in each measurement by a factor of $\sqrt{N}$. For example, if you average together N=100 measurements, the uncertainty in the average will be $\sqrt{N} = \sqrt{100} = 10$ times smaller than the uncertainty in one measurement.

Propagation of Uncertainty

In this course you will often need to plug measured values into an equation so that you can determine some other quantity. For example, if you measure the position of an object at different times, you could use this data to calculate the object's velocity. Since your measurements had uncertainty, your calculation that used the measurements will also have an uncertainty. The process of determining the uncertainty in a calculated value is called **propagation of uncertainty** or "**propagation of errors**".

Unfortunately the general procedure for propagating uncertainty involves calculus, which is beyond the scope of this course. Instead, we will simply give the results you need to use for the cases you will encounter in the labs.

As a first example, suppose that you have measured the distance x and object travelled in a measured time interval t. From these two measurements, you find the velocity v using:

$$v = x / t, \qquad [4]$$

If the uncertainty in x is δx and uncertainty in t is δt, then the uncertainty δv in the velocity v is:

$$\delta v = v\sqrt{\left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta t}{t}\right)^2} \qquad [5]$$

From this equation, you can also define the **fractional uncertainty** or **relative uncertainty**:

$$\frac{\delta v}{v} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta t}{t}\right)^2} \qquad [5]$$

As a second example, consider a situation where a parameter is raised to a power in the calculation. For example, suppose that:

$$x = \frac{1}{2} g t^2 \qquad [6]$$

Notice here that the time t is raised to the power 2. If the uncertainty in t is δt and uncertainty in g is δg, then the uncertainty δx in the position x is:

$$\delta x = x\sqrt{\left(2\frac{\delta t}{t}\right)^2 + \left(\frac{\delta g}{g}\right)^2} \qquad [7]$$

Note here that there is a factor of 2 in front of the δt/t term. This factor of 2 came from the t^2 factor in Equation [6].

V. Lab Exercise: Measuring Your Reaction Time

In this exercise, you and your partner will work together to measure your reaction time. To do this you will catch a ruler that has been dropped by your partner, as shown in Figure 1. If your reflexes are slow, the ruler will drop farther than if your reflexes are fast. The distance the ruler falls before you catch it will be recorded for several trials. From this data, you will calculate the average distance the ruler falls before it is caught and the uncertainty in this average distance. From the average distance, you can then find your reaction time.

A. Collect data on your reaction time

1. Have your partner hold the ruler vertically at a comfortable height near you. Place your own open hand just below the ruler such that it would pass through your fingers and thumb if dropped, as seen in the left panel of Figure 1. Make sure that the zero mark of the ruler is even with the uppermost extent of the sides of your fingers.

2. Have the partner drop the ruler without warning. Close your fingers and thumb to catch the ruler as quickly as you can react to it.

3. Record in a spreadsheet the position *x* on the ruler flush with the uppermost extent of your fingers. Measure x to the nearest millimeter.

4. Repeat steps 1-3 nine more times, for a total of ten trials.

5. Trade roles with your partner and repeat steps 1-4. In the end, you will have ten measurements of your reaction time and your partner's reaction time.

B. Computing the Average Distance and its Uncertainty

1. Calculate the average distance x_{avg} from your data using Eq. (1). You can (and should) use Excel's built-in "SUM" function for the addition, but just this once, **do NOT** use the built-in "AVERAGE" function. Instead, just this one time build the functions manually to see first-hand how they work. If you are still not comfortable with EXCEL, turn back to Experiment 1 or get help from your partner, a classmate, or the TA. Each partner should do this for his or her own data.

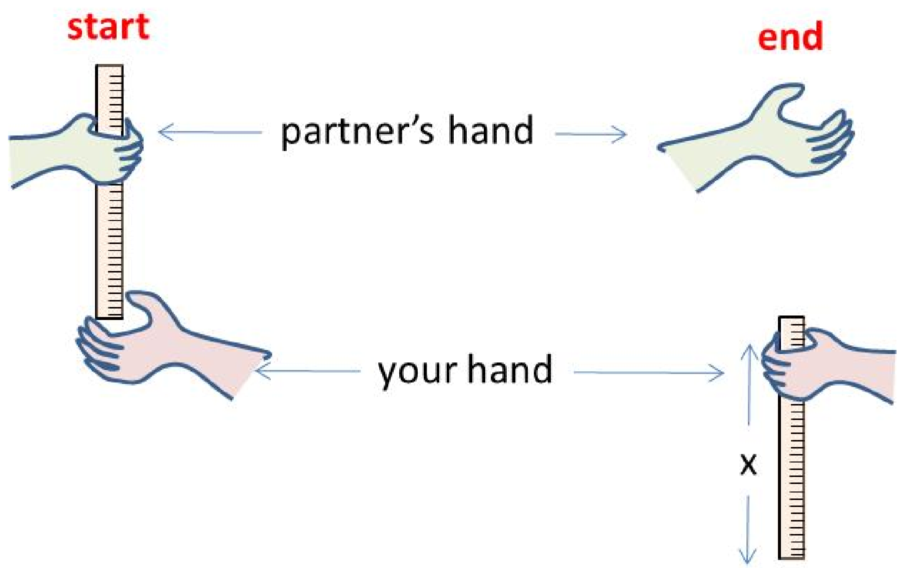

Figure 1

2. Now use EXCEL's "AVERAGE" function to find x_{avg}. Check that this is the same as you found in the previous step. Each partner should do this for his or her own data.

3. Compute the standard deviation σ_x of your measurements of x using Eq. (2). The easiest way to do this is to make a column with the squared differences $(x_i-x_{avg})^2$ in it. After you make this column, you can sum it up, divide by N-1, and then take the square root to get the standard deviation of your measurements of x. Each partner should do this for his or her own data.

4. Now use EXCEL's "STDEV" function to find the standard deviation σ_x of your measurements of x. Check that this is the same as you found in the previous step. Each partner should do this for his or her own data.

5. Calculate the uncertainty δx_{avg} in the average distance x_{avg} using Eq. (3). Each partner should do this for his or her own data.

C. Analysis: Extracting Your Reaction Time from your Data

1. When an object falls under the influence of gravity, the distance it falls *x* is related to the time of flight by the kinematic relationship

$$x = \frac{1}{2} g t^2 \qquad [8]$$

where g = 9.80102 m/s^2 is the acceleration due to gravity at the location of the lab. In part A, you measured the average distance x_{avg} that the ruler fell during your own reaction time. To determine your average reaction time corresponding to the measured distance x_{avg}, you can solve Eq. (8) for t and get:

$$t = \sqrt{2x/g} \qquad [9]$$

Using your value for x_{avg} and the accepted value for g of 9.80102 m/s^2, calculate your reaction time t_{avg} in your spreadsheet. Each partner should do this for his or her own data.

2. To find the uncertainty δt_{avg} in your reaction time t_{avg}, it is possible to start from Equation [9] and propagate the uncertainties in x and g. Since **both** y and g are raised to the ½ power in Equation [9], it can be shown that one finds:

$$\delta t_{avg} = t_{avg}\sqrt{\left(\frac{\delta x_{avg}}{2x_{avg}}\right)^2 + \left(\frac{\delta g}{2g}\right)^2} \qquad [10]$$

Here you should use the accepted value g = 9.80102 m/s^2 and its uncertainty δg = 0.00004 m/s^2. Use Equation [10] to find the uncertainty δt_{avg} in your reaction time t_{avg}.

3. You should have noticed that the uncertainty in the accepted value for g was very small compared to g. This value of g was determined by a group at the National Bureau of Standards that did the measurement on the third floor of the physics building using special equipment, so you should not be too surprised that they achieved such a precise result. Repeat the calculation of δt_{avg} without the contribution from the acceleration of gravity g. How does your result compare with the value of δt_{avg} calculated in the previous step?

In this case, the contribution of the uncertainty in y is *dominant* over the uncertainty in g. If you compare the value of $\delta g/g$ to $\delta x_{avg}/x_{avg}$, you should notice that the former is much smaller than the latter.

Similar cases will arise throughout the semester, and in every instance, it will be acceptable (and encouraged) to ignore the negligible contribution(s) and perform the uncertainty calculation using only the dominant contribution(s).

D. Finishing Up

- Make sure you get your lab partner's name and his or her contact information.

- Make sure you save a copy of your spreadsheet on a memory stick or e-mail yourself a copy.

- In addition, don't forget that you need to prepare a Lab Report for this Experiment and it must be submitted before it is due (see the syllabus). Also, make sure to review the Guidelines for Writing a Lab Report, right after the main introduction in this Lab Manual.

- Finally, don't forget to prepare for the next lab and submit your answers to the prelab questions for the next lab before they are due.