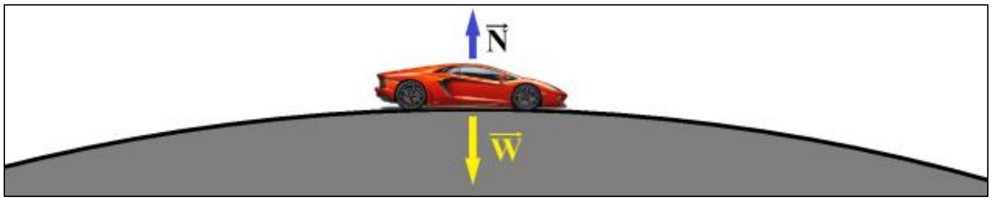
Lecture 9: Gravity, Work, and Energy

Physics for Engineers & Scientists (Giancoli): Chapters 5, 6 and 7
University Physics V1 (Openstax): Chapters 6, 7, and 13

Example: A car drives over a hill. At the crest, the radius of curvature is 50.0 m. What is the maximum speed the car can have and still keep its tires on the road?



Step 1: Which force(s) is acting as F_C?

The centripetal force must be directed vertically downward (towards the center of the circle). As there are two vertical forces acting on the car, these must combine to act as the centripetal force.

$$F_C = W - N$$

Step 2: Solve for F_C: $F_C = W - N = mg - N$

To determine the maximum velocity, we need the maximum centripetal force. W=mg is fixed in value. The normal force can take any positive value. The maximum value of F_C comes when N=0.

$$F_C = mg$$

Step 3: Set $F_C = mv^2/r$: $F_C = mg = \frac{mv^2}{r}$ $\quad g = \frac{v^2}{r}$ $\quad v^2 = gr$

$$v = \sqrt{gr} = \sqrt{\left(9.80\frac{m}{s^2}\right)(50.0\ m)} = 22.1\frac{m}{s}$$

The Forces of Nature

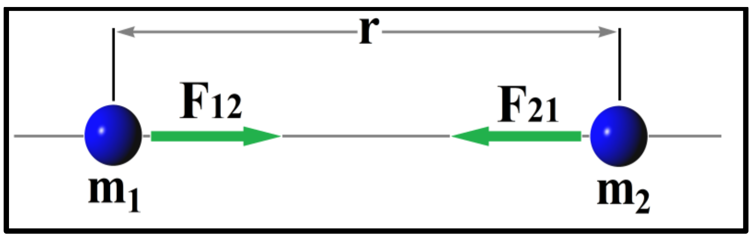
- The **Gravitational Force** creates forces between massive objects. It is responsible for pulling objects to the surface of the Earth, causing planets to orbit the sun, and causing stars to orbit the centers of galaxies.
- The **Electromagnetic Force** creates forces between electric charges. It allows us to generate and use electrical power, transmit signals across vast distances, and use a multitude of convenient devices.
- The **Weak Nuclear Force** is most noticeable in radioactive decays, but also plays a role in nuclear fission (as used in nuclear power plants and nuclear bombs).
- The **Strong Nuclear Force** binds quarks into hadrons (such as protons and neutrons). In addition, it holds atomic nuclei together.

Newton's Law of Gravity

- Newton's Law of Gravity determines the magnitude of the gravitational force (F) between two objects.

$$F = G\frac{m_1 m_2}{r^2}$$

  - F is the force on both massive objects (equal and opposite)
  - G is the gravitational constant $\left(G = 6.673 \times 10^{-11}\ \frac{N \cdot m^2}{kg^2}\right)$
  - m_1 and m_2 are the masses of the two objects
  - r is the distance separating the two object's centers of mass.
- This force is attractive, and is directed along the line connecting the two object's centers of mass.



- The gravitational acceleration (g) is caused by the Earth's gravitational pull and can be directly calculated from Newton's law of gravity, given the mass of the Earth ($M_E = 5.98 \times 10^{24}$ kg), the radius of the earth ($R_E = 6.38 \times 10^6$ m) and the gravitational constant ($G = 6.673 \times 10^{-11}\ \frac{N \cdot m^2}{kg^2}$).

$$W = G\frac{M_E m}{R_E^2} = m\left[\frac{GM_E}{R_E^2}\right] = mg$$

$$g = \frac{GM_E}{R_E^2} = \frac{\left(6.673 \times 10^{-11}\ \frac{N \cdot m^2}{kg^2}\right)\left(5.98 \times 10^{24}\ kg\right)}{(6.38 \times 10^6\ m)^2} = 9.8035\frac{m}{s^2}$$

Satellites in Circular Orbits

- For orbiting satellites, gravitational attraction acts as the centripetal force. Let's let the mass of a satellite be m_s and the mass of the Earth be M_E.

$$F_C = F_g \qquad \frac{m_s v^2}{r} = G\frac{m_s M_E}{r^2} \qquad \frac{v^2}{r} = G\frac{M_E}{r^2} \qquad v^2 = G\frac{M_E}{r} \qquad v = \sqrt{G\frac{M_E}{r}}$$

- Each orbital radius can only be held by an object with a specific velocity, and that velocity is independent of that object's mass:

$$v = \sqrt{G\frac{M_E}{r}}$$

Example: The moon orbits the Earth at a distance of 3.85 x 10^8 m. The Earth's mass is 5.98 x 10^{24} kg. Find the period for the moon's motion around the Earth (in days).

$$v = \sqrt{G\frac{M_E}{r}} = \sqrt{\left(6.673 \times 10^{-11}\frac{Nm^2}{kg^2}\right)\frac{(5.98 \times 10^{24}\ kg)}{(3.85 \times 10^8\ m)}} = 1018.077\frac{m}{s}$$

$$v = \frac{2\pi r}{T} \qquad T = \frac{2\pi r}{v} = \frac{2\pi(3.85 \times 10^8\ m)}{1018.077\frac{m}{s}} = 2{,}376{,}073\ s\left(\frac{1\ hour}{3600\ s}\right)\left(\frac{1\ day}{24\ hours}\right) = 27.5\ days$$

Example: Objects orbiting the center of galaxy M87 have been found to have an orbiting speed of 7.5 x 10^5 m/s for matter orbiting at a distance of 5.7 x 10^{17} m from the center. Find the mass (M) of the object located at the center.

$$v = \sqrt{G\frac{M_{obj}}{r}} \qquad v^2 = G\frac{M_{obj}}{r} \qquad v^2 r = GM_{obj}$$

$$M_{obj} = \frac{v^2 r}{G} = \frac{\left(7.5 \text{ x } 10^5\ \frac{m}{s}\right)^2 (5.7 \text{ x } 10^{17}\ m)}{6.673 \times 10^{-11}\frac{Nm^2}{kg^2}} = 4.8 \times 10^{39}\ kg$$

As the mass of our sun is 2.0 $\times 10^{30}$ kg, the object at the center of M87 is 2.4 billion solar masses. Due to the limited volume available for this amount of mass it was concluded that the object at the center of M87 was a black hole.

Note: Because of the equivalence of inertial mass and gravitational mass, Einstein concluded that there should be no difference between gravity and an accelerating reference frame. This is known as "the equivalence principle", and it led him to conclude that light should bend in a gravitational field even though it has no mass.

Artificial Gravity

- For accelerating reference frames we can account for the acceleration by adding the effects in with gravity, creating "apparent gravity". We used a_F as the acceleration of the reference frame.

$$\vec{g}_{app} = \vec{g} - \vec{a}_F$$

- For an object in orbit around the Earth, the gravitational force acts as the centripetal acceleration. Under these conditions the apparent gravity is zero.

$$\vec{a}_F = \vec{g} \qquad \vec{g}_{app} = \vec{g} - \vec{g} = 0$$

- Astronauts spending long periods in space suffer debilitating conditions due to the lack of gravity, including loss of bone and muscle mass. Artificial gravity provides a means of eliminating these effects, allowing the life forms of Earth to spend an indefinite amount of time in space.

- By making the object in orbit rotate, we can introduce an additional centripetal force that can be used to create artificial gravity. This gravity is equal to a_C, but pointing radially outward.

$$\vec{a}_F = \vec{g} + \vec{a}_C \qquad \vec{g}_{app} = \vec{g} - \vec{a}_F = \vec{g} - (\vec{g} + \vec{a}_C) = -\vec{a}_C$$

Example: The outer wall of a toroidal (donut-shaped) space station is 50.0 m from its central axis. If the toroid rotates uniformly to create a gravity equivalent to Earth's surface, what is the speed of the outer wall?

$$a_c = g = \frac{v^2}{r} \qquad \text{v} = \sqrt{\text{gr}} = \sqrt{\left(9.80\,\frac{\text{m}}{\text{s}^2}\right)(50.0\text{ m})} = 22.1\,\frac{\text{m}}{\text{s}}$$

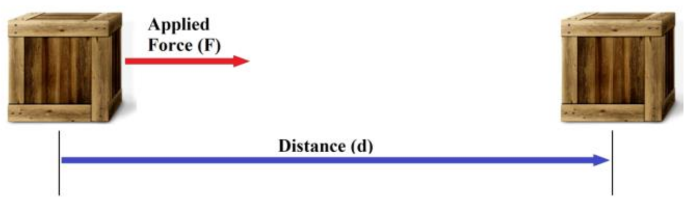
Work and Energy

- All motion and interactions can be understood in terms of energy and the exchange of energy.
- Work is derived from force (a vector), but work and energy are both scalar quantities (not vectors!)
- Many of the problems you have been working can be solved using an energy-based approach.
- In most cases, an energy-based approach to solving problems is preferable to other means.

In other words, don't go running back to kinematics on your homework!

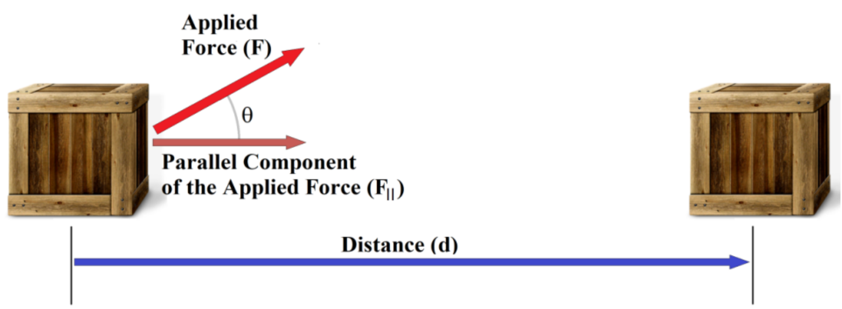
Work $\quad W = \vec{F} \cdot \vec{d}$

- When an object moves a distance d in the direction of an applied force, the work done by that force is the product of the force and the distance.



$$\text{W} = \text{F}\cdot\text{d}$$

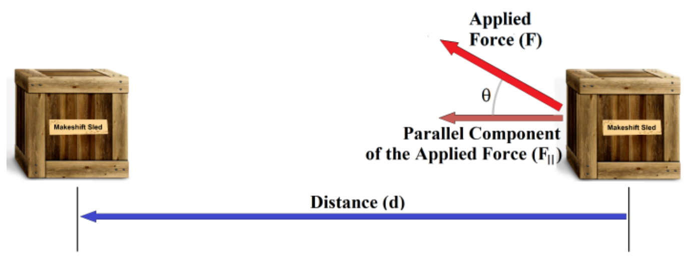
- The standard unit of work and energy the Joule. $\quad 1\text{ J} = 1\text{ N}\cdot\text{m}$
- When an object moves a distance d NOT in the direction of an applied force, the work done is the product of the parallel component of the force and the distance.



$$\text{W} = \text{F}_{\parallel}\cdot\text{d} = \text{F}\cdot\text{d}\cdot\text{Cos}(\theta) = \vec{\text{F}}\cdot\vec{\text{d}}$$

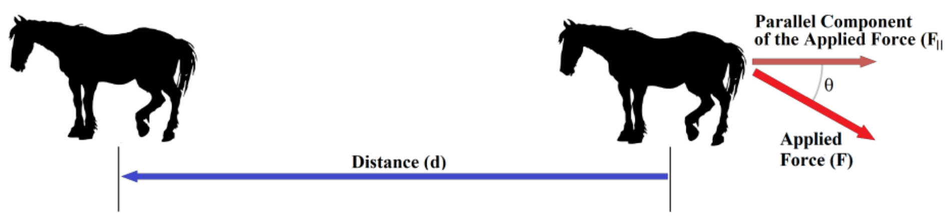
- This is also equivalent to multiply the applied force in full by the component of the distance in the direction of that force.

<u>Example</u>: Two horses pull a man on a makeshift sled. The man and the sled have a combined mass of 204 kg, and the force of friction between the sled and the ground is 700 N. When the horses pull the sled, each of the three chains has a tension of 396 N and makes an angle of 30.0° with respect to the horizontal as they pull the man a distance of 20.2 m. Determine A) the work done on the sled by one of the chains, B) the work done on the horses by one of the chains, and C) the work done on the sled by friction.



A) $W = F_{\parallel} \cdot d = F \cdot d \cdot \text{Cos}(\theta) = (396\ \text{N})(20.2\ \text{m})\text{Cos}(30.0°) = 6.93\ \text{kJ}$

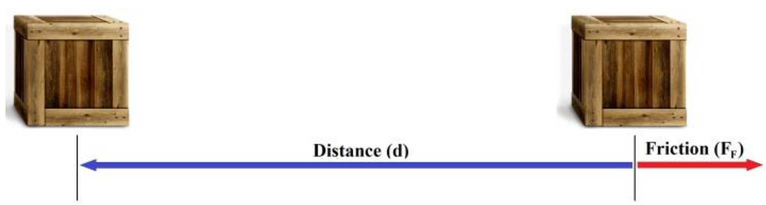
The positive sign on W indicates that the sled is gaining energy.



B) The angle between the force and the distance is now 180° ± θ

$$W = F_{\parallel} \cdot d = F \cdot d \cdot \text{Cos}(180° \pm \theta) = (396\ \text{N})(20.2\ \text{m})\text{Cos}(210.0°) = -6.93\ \text{kJ}$$

The negative sign on W indicates that the horse is losing energy.
The energy lost by the horse is being given to the sled.



A) The angle between the force and the distance is now 180°

$$W = F_{\parallel} \cdot d = F \cdot d \cdot \text{Cos}(180°) = (700\ \text{N})(20.2\ \text{m})\text{Cos}(180°) = -14.14\ \text{kJ}$$

The remaining energy absorbed by the sled is converted to motion of the sled.
Three chains each deliver 6.93 kJ and friction removes 14.14 kJ
Change in Energy = 3(6.93 kJ) – 14.14 kJ = 6.65 kJ

As friction always opposes motion, any work done by friction to a moving object will always be negative.

Example: How much work is done by a constant force, F = (3.26 N)**î** + (5.67 N)**ĵ**, as it acts on an object that moves from point P_1 = (1.23 m, 4.15 m) to point P_2 = (2.71 m, 3.85 m)?

$$\vec{d} = \Delta x\hat{\boldsymbol{\imath}} + \Delta y\hat{\boldsymbol{\jmath}} = (x_2 - x_1)\hat{\boldsymbol{\imath}} + (y_2 - y_1)\hat{\boldsymbol{\jmath}}$$

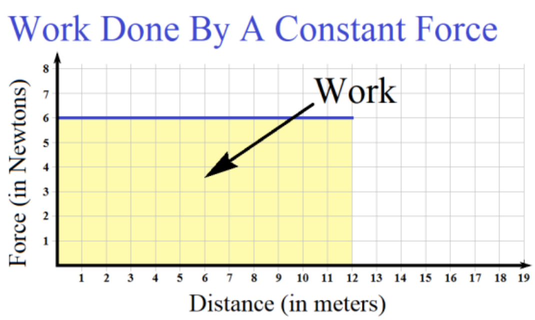
$$\vec{d} = (2.71\,m - 1.23m)\hat{\boldsymbol{\imath}} + (3.85\,m - 4.15m)\hat{\boldsymbol{\jmath}} = (1.48\,m)\hat{\boldsymbol{\imath}} + (-0.30\,m)\hat{\boldsymbol{\jmath}}$$

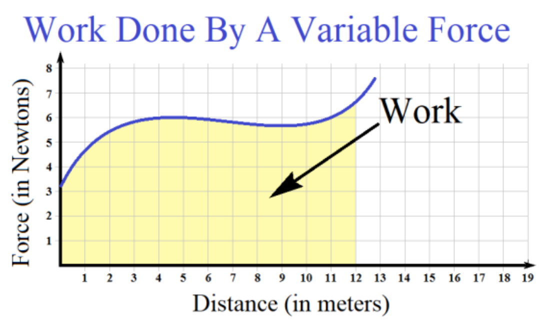
$$W = \vec{F}\cdot\vec{\mathrm{d}} = \{(3.26\,N)\hat{\boldsymbol{\imath}} + (5.67\,N)\hat{\boldsymbol{\jmath}}\}\cdot\{(1.48\,m)\hat{\boldsymbol{\imath}} + (-0.30m)\hat{\boldsymbol{\jmath}}\}$$

$$W = (3.26\,N)(1.48\,m) + (5.67\,N)(-0.30m) = 3.12\,J$$

Work for Variable Forces

- For a constant force, the work may be considered to be the area created on a plot of force versus distance.
- The same holds true for a variable force. The work is the area under the curve when force is plotted against distance.





- In one dimensional motion: $W = \int_{x_1}^{x_2} F(x)dx$
- In higher dimensional motion: $W = \int_{P_1}^{P_2} \vec{F}\cdot d\vec{r}$
  - W is the work done
  - P_1 is the starting point
  - P_2 is the ending point
  - $\vec{F}$ is the force vector, which is a function of position.
  - $d\vec{r}$ is an infinitesimal displacement vector
  - $\vec{F}\cdot d\vec{r}$ is a vector dot-product

Example: An object moves from the origin to x = 1.53 m under the influence of a single force given by F(x) = (2.54 N/m^2)x^2 + (7.95 N/m)x + (16.95 N). Determine the work done by the force.

$$W = \int_{x_1}^{x_2} F(x)dx = \int_0^{1.53\,m} (\alpha x^2 + \beta x + \gamma)dx = \left\{\frac{\alpha}{3}x^3 + \frac{\beta}{2}x^2 + \gamma x\right\}_0^{1.53\,m}$$

$$W = \frac{\left(2.54\,\frac{N}{m^2}\right)}{3}(1.53\,m)^3 + \frac{\left(7.95\frac{N}{m}\right)}{2}(1.53\,m)^2 + (16.95\,N)(1.53\,m) = 38.3\,J$$

Example: An object moves from the origin to point P = (1.12 m, 1.74 m) under the influence of a single force F(x,y) = {(3.17 N/m)x}î + {(1.15 N/m^2)y^2 + (9.41 N)}ĵ. Determine the work done by the force.

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} = \int_{P_1}^{P_2} (F_x\hat{\imath} + F_y\hat{\jmath}) \cdot (dx\hat{\imath} + dy\hat{\jmath}) = \int_{x_1}^{x_2} F_x dx + \int_{y_1}^{y_2} F_y dy$$

$$W = \int_0^{1.12m} \alpha x dx + \int_0^{1.74\,m} \{\beta y^2 + \gamma\} dy = \left\{\frac{\alpha}{2}x^2\right\}_0^{1.12\,m} + \left\{\frac{\beta}{3}y^3 + \gamma y\right\}_0^{1.74\,m}$$

$$W = \frac{\left(3.17\frac{N}{m}\right)}{2}(1.12\,m)^2 + \frac{\left(1.15\frac{N}{m^2}\right)}{3}(1.74\,m)^3 + (9.41\,N)(1.74\,m) = 20.4\,J$$

Kinetic Energy $\quad KE = \frac{1}{2}mv^2$

- **Kinetic energy** is the energy an object possesses due to its motion.

$$W = \int_{x_1}^{x_2} F(x)dx = \int_{x_1}^{x_2} ma \cdot dx = \int_{x_1}^{x_2} m \cdot \frac{dv}{dt} \cdot dx = \int_{x_1}^{x_2} m \cdot \frac{dx}{dt} \cdot dv$$

Remember, dv and dx are just numbers and can be swapped, infinitesimally small, but numbers nonetheless.

$$W = \int_{v_1}^{v_2} mv \cdot dv = \left\{\frac{1}{2}mv^2\right\}_{v_1}^{v_2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

- We define kinetic energy to be: $KE = \frac{1}{2}mv^2$
- This derivation suggests that any work done to an object results in a change in kinetic energy. This is valid when all the forces acting on an object are accounted for.

$$W = \Delta KE = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

- While this view is conceptually clearer, it is mathematically equivalent to the kinematics done previously (simply multiply an equation by ½m).

$$v^2 = v_0^2 + 2a(x - x_0) \qquad \frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 + ma(x - x_0)$$

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = ma(x - x_0) = Fd = W$$

Example: A 5.00 kg bald eagle is initially gliding horizontally at a speed of 11.3 m/s. It begins flapping its wings, generating a horizontal force of 19.6 N. How fast is the Eagle flying when it stops flapping its wings after a distance of 21.3 m??

$$W = \Delta KE \qquad Fd = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 \qquad 2Fd = mv^2 - mv_0^2 \qquad \frac{2Fd}{m} = v^2 - v_0^2$$

$$v^2 = v_0^2 + \frac{2Fd}{m} \qquad v = \sqrt{v_0^2 + \frac{2Fd}{m}} = \sqrt{\left(11.3\frac{m}{s}\right)^2 + \frac{2(19.6N)(21.3m)}{5.00\,kg}} = 17.2\frac{m}{s}$$