Lecture 8: Vectors & Two-Dimensional Kinematics

Physics for Engineers & Scientists (Giancoli): Chapter 3
University Physics V1 (Openstax): Chapters 2 & 4

Multiplying Two Vectors

- There are two ways to multiply a vector by another vector.
  - One is called a "dot-product", which produces a scalar. ($\vec{A} \cdot \vec{B}$)
  - One is called a "cross-product", which produces a vector. ($\vec{A} \times \vec{B}$)

  We won't be using cross-products for a while. We will return to this when needed.

- There are two ways to calculate a dot product. One uses magnitudes and angles. The other uses components. Choose the more convenient option.

$$\vec{A} \cdot \vec{B} = AB \cdot \mathrm{Cos}\theta = A_xB_x + A_yB_y + A_zB_z$$

  - The angle, θ, is the angle between the two vectors.

$$\mathrm{Cos}\ \theta = \mathrm{Cos}(\theta_A - \theta_B) = \mathrm{Cos}\ (\theta_B - \theta_A)$$

- The square of a vector is considered a dot product with itself. We can represent the magnitude of a vector as a dot product.

$$\vec{A}^2 = \vec{A} \cdot \vec{A} = A_x^2 + A_y^2 + A_z^2$$

$$A = |\vec{A}| = \sqrt{\vec{A}^2} = \sqrt{\vec{A} \cdot \vec{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

- Dot products with unit vectors.
  - The dot product of a unit vector with itself is 1. *(Unit magnitude and $\theta = 0°$)*

$$\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = 1$$

  - The dot product of a unit vector with any other unit vector is 0. *($\theta = 90°$)*

$$\hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{\imath} = \hat{\imath} \cdot \hat{k} = \hat{k} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{k} = \hat{k} \cdot \hat{\jmath} = 0$$

  - With unit vectors we can multiply products term by term to produce the answer.

$$\vec{A} \cdot \vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath}) \cdot (B_x\hat{\imath} + B_y\hat{\jmath})$$

$$\vec{A} \cdot \vec{B} = A_xB_x \underbrace{\hat{\imath} \cdot \hat{\imath}}_{1} + A_xB_y \underbrace{\hat{\imath} \cdot \hat{\jmath}}_{0} + A_yB_x \underbrace{\hat{\jmath} \cdot \hat{\imath}}_{0} + A_yB_y \underbrace{\hat{\jmath} \cdot \hat{\jmath}}_{1}$$

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y$$

Example: Find the dot product of $\vec{A} = 3\hat{\imath} + 4\hat{\jmath}$ and $\vec{B} = 5\hat{\imath} + 12\hat{\jmath}$.

We can do this two ways. In this case, the first (using components) is simplest.

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y = 3 \cdot 5 + 4 \cdot 12 = 15 + 48 = 63$$

The second method requires determining the magnitudes and angles of both vectors.

$$A = |\vec{A}| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$\theta_A = \mathrm{Tan}^{-1}\left(\frac{A_y}{A_x}\right) = \mathrm{Tan}^{-1}\left(\frac{4}{3}\right) = 53.13°$$

$$B = |\vec{B}| = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

$$\theta_B = \mathrm{Tan}^{-1}\left(\frac{B_y}{B_x}\right) = \mathrm{Tan}^{-1}\left(\frac{12}{5}\right) = 67.38°$$

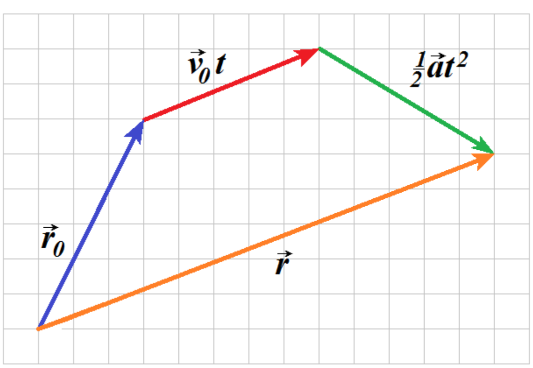
$$\theta_A - \theta_B = 53.13° - 67.38° = -14.25°$$

$$\vec{A} \cdot \vec{B} = AB \cdot \mathrm{Cos}\theta = 5 \cdot 13 \cdot \mathrm{Cos}(14.25°) = 63$$

Two-Dimensional Quantities *(Everything but time (t) is now a vector).*

Quantity	One Dimension	Two Dimensions
Position	x (or y)	$\vec{r} = x\hat{\imath} + y\hat{\jmath}$
Initial Position	x_0 (or y_0)	$\vec{r}_0 = x_0\hat{\imath} + y_0\hat{\jmath}$
Displacement	Δx (or Δy)	$\Delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath}$
Average Velocity	$v_{avg} = \frac{\Delta x}{\Delta t}$	$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{\imath} + \frac{\Delta y}{\Delta t}\hat{\jmath} = v_{x-avg}\hat{\imath} + v_{y-avg}\hat{\jmath}$
Average Acceleration	$a_{avg} = \frac{\Delta v}{\Delta t}$	$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\Delta v_x}{\Delta t}\hat{\imath} + \frac{\Delta v_y}{\Delta t}\hat{\jmath} = a_{x-avg}\hat{\imath} + a_{y-avg}\hat{\jmath}$
Const a equation #1 (no x)	$v = v_0 + at$	$\vec{v} = \vec{v}_0 + \vec{a}t$
Const a equation #2 (no a)	$x = x_0 + \frac{1}{2}(v + v_0)t$	$\vec{r} = \vec{r}_0 + \frac{1}{2}(\vec{v} + \vec{v}_0)t$
Const a equation #3 (no v)	$x = x_0 + v_0 t + \frac{1}{2}at^2$	$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$
Const a equation #4 (no t)	$v^2 = v_0^2 + 2a(x - x_0)$	$\vec{v}^2 = \vec{v}_0^2 + 2\vec{a} \cdot (\vec{r} - \vec{r}_0)$

Solving Problems



- If the vectors are all in the same direction, then you can treat it 1-dimensionally.
- If not, then you split everything into components and solve x and y separately, then recombine back into vectors at the end.
- The x and y equations are linked by time and/or any angles given. Apart from that, the x- and y-components are independent from each other.
- Projectile motion problems (i.e. falling near the Earth's surface): $a_{horizontal} = 0$, $a_{vertical} = -9.80\ m/s^2$ (downward)

Example: A player kicks a ball at rest. The ball remains in contact with the kicker's foot for 0.0500s, during which time it experiences an acceleration of 340.0m/s². The ball is launched at an angle of 51.0° above the ground. Determine the horizontal and vertical components of the launch velocity.

$$\vec{v}_0 = 0 \qquad t = 0.0500\ s \qquad \vec{a} = 340.0\angle 51.0° \qquad \vec{v} = ?$$

$$\vec{v} = \vec{v}_0 + \vec{a}t = \vec{a}t = \left(340.0\,\frac{m}{s^2}\right)(0.0500\ s) = 17.0\,\frac{m}{s}$$

$$v_x = v \cdot \mathrm{Cos}(\theta) = \left(17.0\,\frac{m}{s}\right)\mathrm{Cos}(51.0°) = 10.698\,\frac{m}{s} \Rightarrow 10.7\,\frac{m}{s}$$

$$v_y = v \cdot \mathrm{Sin}(\theta) = \left(17.0\,\frac{m}{s}\right)\mathrm{Sin}(51.0°) = 13.211\,\frac{m}{s} \Rightarrow 13.2\,\frac{m}{s}$$

<u>Example</u>: For the previous example, after the ball leaves the kicker's foot, how far from the initial position will it land?

x-components: $x_0 = 0$ $v_{0x} = 10.698$ m/s $a_x = 0$ $v_x = v_{0x}$

As a_x=0, the only equation available is: $x = x_0 + v_x t = v_x t$

This would give us the answer ...if we had t $\rightarrow$ Need t from y-components.

y-components: What is v_y when it lands?

When $y = y_0$, then $v = -v_0$. $\{\, v_y^2 = v_{oy}^2 + 2a_y \underbrace{(y - y_0)}_{0} \rightarrow v_y^2 = v_{0y}^2 \,\}$

$y_0 = y = 0$ $v_{oy} = 13.211$ m/s $v_y = -\,v_{oy} = -13.211$ m/s $a_y = -g = -9.80$ m/s^2 t = ???

$$v_y = v_{0y} + a_y t \qquad v_y - v_{0y} = a_y t$$

$$t = \frac{v_y - v_{0y}}{a_y} = \frac{v_y - v_{0y}}{-g} = \frac{-v_{0y} - v_{0y}}{-g} = \frac{-2v_{0y}}{-g} = \frac{2v_{0y}}{g} = \frac{2\left(13.211\,\frac{m}{s}\right)}{9.80\ m/s^2} = 2.6961\ s$$

$$x = v_{0x}t = \left(10.698\,\frac{m}{s}\right)(2.6961\ s) = 28.843\ m \Rightarrow 28.8\ m.$$

<u>Example</u>: For the previous examples, after the ball leaves the kicker's foot, how high will the ball go? (i.e. determine the maximum height)

What do we know about maximum displacement? $v_y = 0$

$y_0 = 0$ $v_{oy} = 13.211$ m/s $v_y = 0$ $a_y = -g = -9.80$ m/s^2 y = ??? (no t)

$$v_y^2 = v_{oy}^2 + 2a_y(y - y_0) \qquad 0 = v_{oy}^2 + 2a_y y \qquad -v_{oy}^2 = 2a_y y$$

$$y = \frac{-v_{oy}^2}{2a_y} = \frac{-v_{oy}^2}{-2g} = \frac{v_{oy}^2}{2g} = \frac{\left(13.211\,\frac{m}{s}\right)^2}{2\left(9.80\,\frac{m}{s^2}\right)} = 8.90462\ m \Rightarrow 8.90\ m$$

Note: we could have found t first.

$$v_y = v_{0y} + a_y t \qquad v_y - v_{0y} = a_y t$$

$$t = \frac{v_y - v_{oy}}{a_y} = \frac{-v_{oy}}{-g} = \frac{v_{oy}}{g} = \frac{13.211\frac{m}{s}}{9.80\frac{m}{s^2}} = 1.3481\ s \quad \text{(half the time of the previous example)}$$

$$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2 = 0 + \left(13.211\,\frac{m}{s}\right)(1.3481\ s) + \frac{1}{2}\left(-9.80\,\frac{m}{s^2}\right)(1.3481\ s)^2 = 8.90462\ m$$