Lecture 5: Vectors and Two-Dimensional Kinematics

Physics for Engineers & Scientists (Giancoli): Chapter 3
University Physics V1 (Openstax): Chapter 2 and Chapter 4

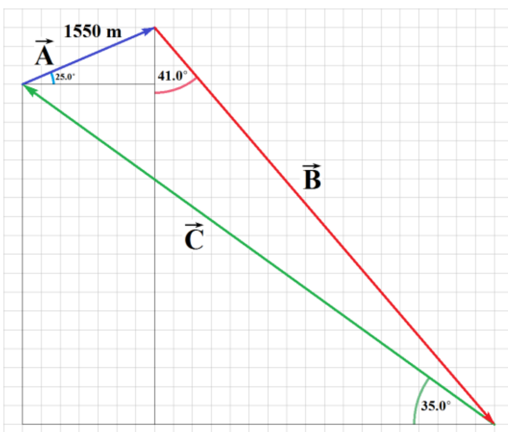
Example: The route followed by a hiker consists of 3 displacement vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$. Vector $\vec{A}$ is along a measured trail and is 1550m in a direction 25.0° north of east. Vector $\vec{B}$ is not along a measured trail but the hiker uses a compass and knows that the direction is 41.0° east of south. Similarly the direction of vector $\vec{C}$ is 35° north of west. The hiker ends up back where she started, so the resulting displacement is zero (or $\vec{A} + \vec{B} + \vec{C} = 0$). Find the magnitude of vector $\vec{B}$ and vector $\vec{C}$.

We can solve this is two different ways. The first option, by using components, requires one to solve a system of two equations. The second option, using the law of sines, requires a good bit of geometry. We will solve this one both ways.

First make a diagram (not easy when most lengths aren't given).

Solution 1 (by components)

Note: The angles given for $\vec{B}$ and $\vec{C}$ are not defined with respect to the positive x-axis. You must make triangles and use trigonometry to find components.



A_x = (1550 m)Cos(25.0°) = 1404.8 m

A_y = (1550 m)Sin(25.0°) = 655.1 m

B_x = B·Sin(41.0°) *{B_x is the opposite side}*

B_y = -B·Cos(41.0°) *{B_y is pointed down}*

C_x = -C·Cos(35.0°) *{C_x is pointed left}*

C_y = C·Sin(35.0°)

$$A_x + B_x + C_x = 0$$

$$1404.8\text{ m} + B\cdot\text{Sin}(41.0°) - C\cdot\text{Cos}(35.0°) = 0$$

$$A_y + B_y + C_y = 0$$

$$655.1\text{ m} - B\cdot\text{Cos}(41.0°) + C\cdot\text{Sin}(35.0°) = 0$$

2 equations, 2 unknowns (B and C).

Multiply the first equation by Sin (35.0°), multiply the second equation by Cos (35.0°), and then add he two equations. The C terms will disappear.

$$(1404.8\text{ m})\cdot\text{Sin}(35.0°) + B\cdot\text{Sin}(41.0°)\cdot\text{Sin}(35.0°) - C\cdot\text{Cos}(35.0°)\cdot\text{Sin}(35.0°) = 0$$

$$\underline{(655.1\text{ m})\cdot\text{Cos}(35.0°) - B\cdot\text{Cos}(41.0°)\cdot\text{Cos}(35.0°) + C\cdot\text{Cos}(35.0°)\cdot\text{Sin}(35.0°) = 0}$$

$$(1404.8\text{ m})\cdot\text{Sin}(35.0°) + (655.1\text{ m})\cdot\text{Cos}(35.0°) + B\cdot\text{Sin}(41.0°)\cdot\text{Sin}(35.0°) - B\cdot\text{Cos}(41.0°)\cdot\text{Cos}(35.0°) = 0$$

$$(1404.8\text{ m})\cdot\text{Sin}(35.0°) + (655.1\text{ m})\cdot\text{Cos}(35.0°) = B\cdot\text{Cos}(41.0°)\cdot\text{Cos}(35.0°) - B\cdot\text{Sin}(41.0°)\cdot\text{Sin}(35.0°)$$

$$(1404.8\text{ m})\cdot\text{Sin}(35.0°) + (655.1\text{ m})\cdot\text{Cos}(35.0°) = \text{B}\cdot\{\text{Cos}(41.0°)\cdot\text{Cos}(35.0°) - \text{Sin}(41.0°)\cdot\text{Sin}(35.0°)\}$$

$$\text{B} = \{(1404.8\text{ m})\cdot\text{Sin}(35.0°) + (655.1\text{ m})\cdot\text{Cos}(35.0°)\}/\{\text{Cos}(41.0°)\cdot\text{Cos}(35.0°) - \text{Sin}(41.0°)\cdot\text{Sin}(35.0°)\}$$

$$\text{B} = \{805.76\text{ m} + 536.63\text{ m}\}/\text{Cos}(76.0°) = 1342.39\text{ m}/\text{Cos}(76.0°) = 5548.86\text{ m} \Rightarrow 5550\text{ m}$$

Solve first equation for C.

$$1404.8\text{ m} + \text{B}\cdot\text{Sin}(41.0°) - \text{C}\cdot\text{Cos}(35.0°) = 0$$

$$1404.8\text{ m} + (5548.86\text{ m})\cdot\text{Sin}(41.0°) - \text{C}\cdot\text{Cos}(35.0°) = 0$$

$$1404.8\text{ m} + 3640.4 = \text{C}\cdot\text{Cos}(35.0°)$$

$$5045.2\text{ m} = \text{C}\cdot\text{Cos}(35.0°)$$

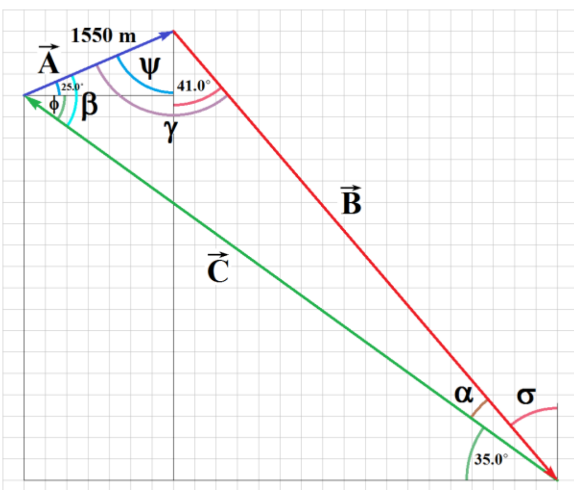
$$\text{C} = 5045.2\text{ m}/\text{Cos}(35.0°) = 6159.1\text{ m} \Rightarrow 6160\text{ m}$$

$$\text{B} = 5550\text{ m} \quad \text{and} \quad \text{C} = 6160\text{ m}$$

Solution 2 (law of sines)

To use the law of sines we need one length (which we have from side A), and all the interior angles. So, first we need to find α, β, and γ.

Start by adding some extra angles to our diagram.



$\beta = 25.0° + \phi = 25.0° + 35.0° = 60.0°$

$\psi = 90° - 25.0° = 65.0°$ *{Interior angles}*

$\gamma = \psi + 41.0° = 65.0° + 41.0° = 106.0°$

$\sigma = 41.0°$ *{Vertical lines are parallel}*

$\alpha = 90° - 35.0° - 41.0° = 14.0°$

Check: $60.0° + 106.0° + 14.0° = 180.0°$

$$\frac{\text{Sin}\,\alpha}{\text{A}} = \frac{\text{Sin}\,\beta}{\text{B}} = \frac{\text{Sin}\,\gamma}{\text{C}}$$

$\text{B}\cdot\text{Sin}(\alpha) = \text{A}\cdot\text{Sin}(\beta)$

$\text{B} = \text{A}\cdot\text{Sin}(\beta)/\text{Sin}(\alpha) = (1550\text{ m})\cdot\text{Sin}(60.0°)/\text{Sin}(14.0°)$

$\text{B} = 5548.65\text{ m} \Rightarrow 5550\text{ m}$

$\text{C}\cdot\text{Sin}(\alpha) = \text{A}\cdot\text{Sin}(\gamma)$

$\text{C} = \text{A}\cdot\text{Sin}(\gamma)/\text{Sin}(\alpha) = (1550\text{ m})\cdot\text{Sin}(106.0°)/\text{Sin}(14.0°)$

$\text{C} = 6158.83\text{ m} \Rightarrow 6160\text{ m}$

$\phi = 35.0°$ *{Horizontal lines are parallel}*

It is usually a good idea to determine what method of solving a problem is the most simple and expedient.

Multiplying Two Vectors

- There are two ways to multiply a vector by another vector.
  - One is called a "dot-product", which produces a scalar. ($\vec{A} \cdot \vec{B}$)
  - One is called a "cross-product", which produces a vector. ($\vec{A} \times \vec{B}$)

We won't be using cross-products for a while. We will return to this when needed.

- There are two ways to calculate a dot product. One uses magnitudes and angles. The other uses components. Choose the more convenient option.

$$\vec{A} \cdot \vec{B} = AB \cdot Cos\theta = A_x B_x + A_y B_y + A_z B_z$$

  - The angle, θ, is the angle between the two vectors.

$$Cos\ \theta = Cos(\theta_A - \theta_B) = Cos\ (\theta_B - \theta_A)$$

- The square of a vector is considered a dot product with itself. We can represent the magnitude of a vector as a dot product.

$$\vec{A}^2 = \vec{A} \cdot \vec{A} = A_x^2 + A_y^2 + A_z^2$$

$$A = |\vec{A}| = \sqrt{\vec{A}^2} = \sqrt{\vec{A} \cdot \vec{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

- Dot products with unit vectors.
  - The dot product of a unit vector with itself is 1. *(Unit magnitude and θ = 0°)*

$$\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = 1$$

  - The dot product of a unit vector with any other unit vector is 0. *(θ =90°)*

$$\hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{\imath} = \hat{\imath} \cdot \hat{k} = \hat{k} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{k} = \hat{k} \cdot \hat{\jmath} = 0$$

  - With unit vectors we can multiply products term by term to produce the answer.

$$\vec{A} \cdot \vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath}) \cdot (B_x\hat{\imath} + B_y\hat{\jmath})$$

$$\vec{A} \cdot \vec{B} = A_x B_x \underbrace{\hat{\imath} \cdot \hat{\imath}}_{1} + A_x B_y \underbrace{\hat{\imath} \cdot \hat{\jmath}}_{0} + A_y B_x \underbrace{\hat{\jmath} \cdot \hat{\imath}}_{0} + A_y B_y \underbrace{\hat{\jmath} \cdot \hat{\jmath}}_{1}$$

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y$$

Example: Find the dot product of $\vec{A} = 3\hat{\imath} + 4\hat{\jmath}$ and $\vec{B} = 5\hat{\imath} + 12\hat{\jmath}$.

We can do this two ways. In this case, the first (using components) is simplest.

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y = 3 \cdot 5 + 4 \cdot 12 = 15 + 48 = 63$$

The second method requires determining the magnitudes and angles of both vectors.

$$A = |\vec{A}| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$\theta_A = Tan^{-1}\left(\frac{A_y}{A_x}\right) = Tan^{-1}\left(\frac{4}{3}\right) = 53.13°$$

$$B = |\vec{B}| = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

$$\theta_B = Tan^{-1}\left(\frac{B_y}{B_x}\right) = Tan^{-1}\left(\frac{12}{5}\right) = 67.38°$$

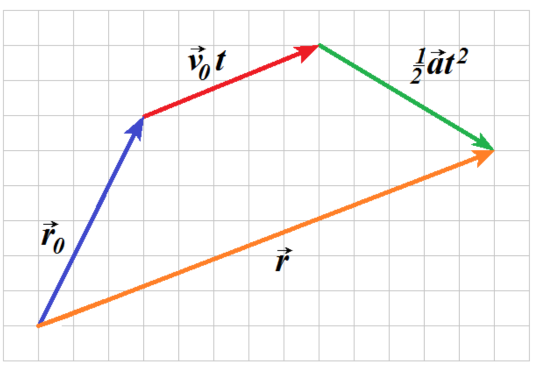
$$\theta_A - \theta_B = 53.13° - 67.38° = -14.25°$$

$$\vec{A} \cdot \vec{B} = AB \cdot Cos\theta = 5 \cdot 13 \cdot Cos(14.25°) = 63$$

Two-Dimensional Quantities *(Everything but time (t) is now a vector).*

Quantity	One Dimension	Two Dimensions
Position	x (or y)	$\vec{r} = x\hat{\imath} + y\hat{\jmath}$
Initial Position	x_0 (or y_0)	$\vec{r}_0 = x_0\hat{\imath} + y_0\hat{\jmath}$
Displacement	Δx (or Δy)	$\Delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath}$
Average Velocity	$v_{avg} = \frac{\Delta x}{\Delta t}$	$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{\imath} + \frac{\Delta y}{\Delta t}\hat{\jmath} = v_{x-avg}\hat{\imath} + v_{y-avg}\hat{\jmath}$
Average Acceleration	$a_{avg} = \frac{\Delta v}{\Delta t}$	$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\Delta v_x}{\Delta t}\hat{\imath} + \frac{\Delta v_y}{\Delta t}\hat{\jmath} = a_{x-avg}\hat{\imath} + a_{y-avg}\hat{\jmath}$
Const a equation #1 (no x)	$v = v_0 + at$	$\vec{v} = \vec{v}_0 + \vec{a}t$
Const a equation #2 (no a)	$x = x_0 + \frac{1}{2}(v + v_0)t$	$\vec{r} = \vec{r}_0 + \frac{1}{2}(\vec{v} + \vec{v}_0)t$
Const a equation #3 (no v)	$x = x_0 + v_0 t + \frac{1}{2}at^2$	$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$
Const a equation #4 (no t)	$v^2 = v_0^2 + 2a(x - x_0)$	$\vec{v}^2 = \vec{v}_0^2 + 2\vec{a}\cdot(\vec{r} - \vec{r}_0)$

Solving Problems



- If the vectors are all in the same direction, then you can treat it 1-dimensionally.
- If not, then you split everything into components and solve x and y separately, then recombine back into vectors at the end.
- The x and y equations are linked by time and/or any angles given. Apart from that, the x- and y-components are independent from each other.
- Projectile motion problems (i.e. falling near the Earth's surface): $a_{horizontal} = 0$, $a_{vertical} = -9.80\ m/s^2$ (downward)

Example: A player kicks a ball at rest. The ball remains in contact with the kicker's foot for 0.0500s, during which time it experiences an acceleration of 340.0m/s². The ball is launched at an angle of 51.0° above the ground. Determine the horizontal and vertical components of the launch velocity.

$$\vec{v}_0 = 0 \quad t = 0.0500\ s \quad \vec{a} = 340.0\angle 51.0° \quad \vec{v} = ?$$

$$\vec{v} = \vec{v}_0 + \vec{a}t = \vec{a}t = \left(340.0\frac{m}{s^2}\right)(0.0500\ s) = 17.0\frac{m}{s}$$

$$v_x = v\cdot \mathrm{Cos}(\theta) = \left(17.0\frac{m}{s}\right)\mathrm{Cos}(51.0°) = 10.698\frac{m}{s} \Rightarrow 10.7\frac{m}{s}$$

$$v_y = v\cdot \mathrm{Sin}(\theta) = \left(17.0\frac{m}{s}\right)\mathrm{Sin}(51.0°) = 13.211\frac{m}{s} \Rightarrow 13.2\frac{m}{s}$$

Example: For the previous example, after the ball leaves the kicker's foot, how far from the initial position will it land?

x-components: $x_0 = 0$ $v_{0x} = 10.698$ m/s $a_x = 0$ $v_x = v_{0x}$

As $a_x=0$, the only equation available is: $x = x_0 + v_x t = v_x t$

This would give us the answer ...if we had t $\rightarrow$ Need t from y-components.

y-components: What is v_y when it lands?

When $y = y_0$, then $v = -v_0$. $\{ v_y^2 = v_{oy}^2 + 2a_y \underbrace{(y - y_0)}_{0} \rightarrow v_y^2 = v_{0y}^2 \}$

$y_0 = y = 0$ $v_{oy} = 13.211$ m/s $v_y = - v_{oy} = -13.211$ m/s $a_y = -g = -9.80$ m/s^2 t = ???

$$v_y = v_{0y} + a_y t \qquad v_y - v_{0y} = a_y t$$

$$t = \frac{v_y - v_{0y}}{a_y} = \frac{v_y - v_{0y}}{-g} = \frac{-v_{0y} - v_{0y}}{-g} = \frac{-2v_{0y}}{-g} = \frac{2v_{0y}}{g} = \frac{2\left(13.211 \frac{m}{s}\right)}{9.80 \text{ m/s}^2} = 2.6961 \text{ s}$$

$$x = v_{0x} t = \left(10.698 \frac{m}{s}\right)(2.6961 \text{ s}) = 28.843 \text{ m} \Rightarrow 28.8 \text{ m.}$$

Example: For the previous examples, after the ball leaves the kicker's foot, how high will the ball go? (i.e. determine the maximum height)

What do we know about maximum displacement? $v_y = 0$

$y_0 = 0$ $v_{oy} = 13.211$ m/s $v_y = 0$ $a_y = -g = -9.80$ m/s^2 y = ??? (no t)

$$v_y^2 = v_{oy}^2 + 2a_y(y - y_0) \qquad 0 = v_{oy}^2 + 2a_y y \qquad -v_{oy}^2 = 2a_y y$$

$$y = \frac{-v_{oy}^2}{2a_y} = \frac{-v_{oy}^2}{-2g} = \frac{v_{oy}^2}{2g} = \frac{\left(13.211 \frac{m}{s}\right)^2}{2\left(9.80 \frac{m}{s^2}\right)} = 8.90462 \text{ m} \Rightarrow 8.90 \text{ m}$$

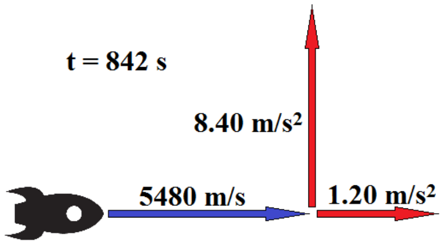
Note: we could have found t first.

$$v_y = v_{0y} + a_y t \qquad v_y - v_{0y} = a_y t$$

$$t = \frac{v_y - v_{oy}}{a_y} = \frac{-v_{oy}}{-g} = \frac{v_{oy}}{g} = \frac{13.211 \frac{m}{s}}{9.80 \frac{m}{s^2}} = 1.3481 \text{ s}$$ (half the time of the previous example)

$$y = y_0 + v_{0y} t + \frac{1}{2} a_y t^2 = 0 + \left(13.211 \frac{m}{s}\right)(1.3481 \text{ s}) + \frac{1}{2}\left(-9.80 \frac{m}{s^2}\right)(1.3481 \text{ s})^2 = 8.90462 \text{ m}$$

Example: A spacecraft is travelling with a velocity of v_{0x} = 5480 m/s along the +x direction. Two engines are turned on for a time of 842 s. One engine gives the spacecraft an acceleration in the +x direction of a_x = 1.20 m/s², while the other gives it an acceleration in the +y direction of 8.40 m/s². At the end of the firing, find (a) v_x and (b) v_y.



t = 842s v_{ox} = 5480m/s v_{oy} = 0 a_x = 1.20m/s² a_y = 8.40m/s²

$$v_x = v_{0x} + a_x t = \left(5480\frac{m}{s}\right) + \left(1.20\frac{m}{s^2}\right)(842\ s) = 6490\frac{m}{s}$$

$$v_y = v_{0y} + a_y t = \left(0\frac{m}{s}\right) + \left(8.40\frac{m}{s^2}\right)(842\ s) = 7070\frac{m}{s}$$

If the quadratic equation gives you trouble, you can find your answer in another way!