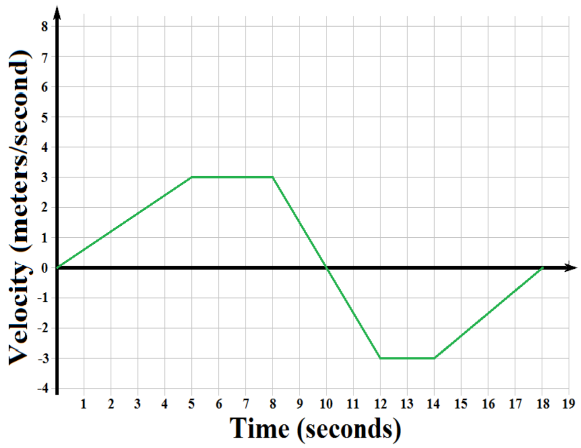
Lecture 4: Vectors

Physics for Engineers & Scientists (Giancoli): Chapter 2
University Physics V1 (Openstax): Chapter 3

Example: From the velocity graph shown determine the following if **x_0=10 m**:

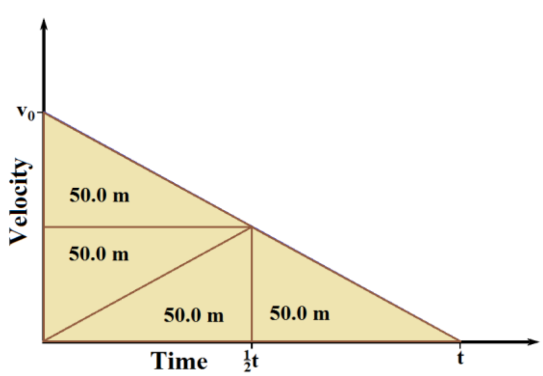


1) What is '**v**' at t = 5?
2) What is '**a**' between 0 s and 5 s?
3) What is '**a**' at t = 6 s?
4) What is '**Δx**' at t = 5s?
5) What is '**x**' at t = 8s?
6) What is '**x**' at t = 14s?

1) $v = 3\ m/s$
2) $a = \frac{\Delta v}{\Delta t} = \frac{v(5s) - v(0s)}{5s - 0s} = \frac{3m/s - 0m/s}{5s} = 0.6\, m/s^2$
3) $a = 0$
4) $\Delta x = ½bh = ½(5s)(3\ m/s) = 7.5\ m$
5) $x = x_0 + \frac{1}{2}bh + LW$ $\qquad x = 10\ m + 7.5\ m + (3s)\left(3\frac{m}{s}\right) = 26.5\ m$
6) $x = x(t = 8) + \Delta x$ $\qquad x = 26.5\ m + \frac{1}{2}bh - \frac{1}{2}bh - LW$

$x = 26.5\ m + 3m - 3m - 6m = 20.5\ m$

Example: A hockey player passes the puck. After uniform deceleration, the puck comes to rest 200.0 m away. How far does it travel in half this time?

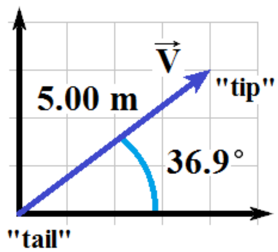


- Draw a graph of velocity
- It starts at v_0 and falls to zero in time t.
- The area under the curve = Δx = 200.0 m
- The distance it covers in half the time is area left of the ½t line.
- We break the area into 4 equal triangles
- The sum of the 4 equal areas must be 200.0 m
- Each triangle is then 50.00 m (a quarter of 200.0 m)
- As 3 triangles make up the area left of the ½t line, the answer is 150.0 m (3 × 50.00 m)

Vectors

- A **vector quantity** is characterized by two properties (magnitude, directions = 2 numbers).
- A **scalar quantity** is characterized by a single property (magnitude = 1 number).

Vectors can have more than two properties/numbers, but (for the moment) we are restricting ourselves to 2-dimensional vectors, which are limited to 2.



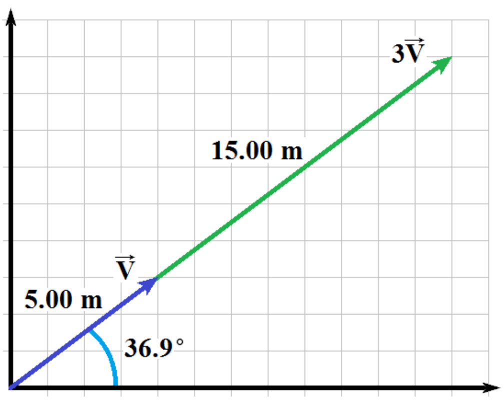
Example Vector:

$\vec{V} = V\angle\theta_V = (5.00\text{ m})\angle 36.9°$

Magnitude (length): $|\vec{V}| = V = (5.00\text{ m})$

Angle: $\theta_V = 36.9°$

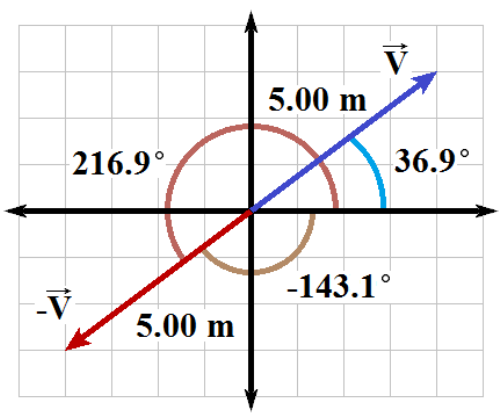
Vector Math



Multiply a Vector by a Positive Scalar

- Multiply the magnitude (V) by the scalar (a)
- $(3.00)\vec{V} = (3.00 \times V)\angle\theta_V =$
  $(3.00)(5.00\text{ m})\angle 36.9° = 15.0\text{ m}\angle 36.9°$

This changes the length without any change to the direction.



Multiply a Vector by "–1"

- Add (or subtract) 180° to (or from) the angle.
- $-1\,\vec{V} = -V\angle\theta_V = V\angle\theta_V \pm 180° =$
  $5.00\text{ m}\angle 216.9°$ or $5.00\text{ m}\angle -143.1°$

Adding 180° and subtracting 180° effectively do the same thing (answers will differ by 360°)

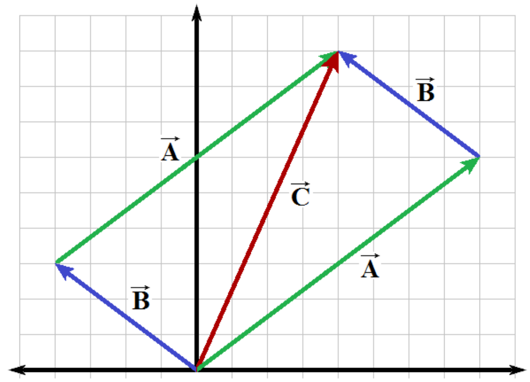
If you turn around, it doesn't matter if you turned to the right or the left. The result is the same.

While these are mathematically equivalent, most homework services will only accept one (they tell you the range of values, typically 0° to 360° or -180° to 180°, that are acceptable).

Multiply a Vector by a Negative Scalar $\Rightarrow$ break it into a "-1" and a positive scalar

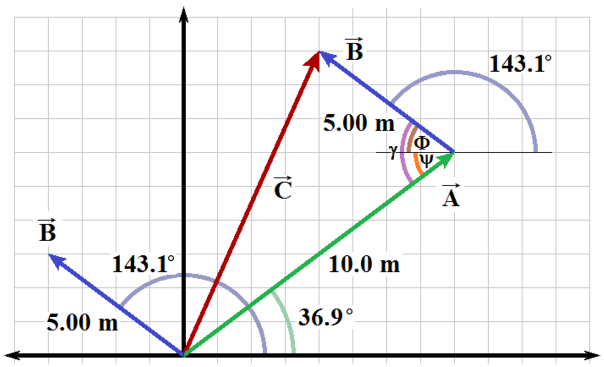
- $(-3.00)\vec{V} = (3.00 \times V)\angle\theta_V \pm 180° = 15.0\text{ m}\angle 216.9°\text{ or } -143.1°$

Adding Vectors



- Leave your first vector where it is.
- Slide the second vector up so its "tail" starts at the "tip" of the first vector.
- The sum of these two vectors is a vector that starts at the tail of the first and goes to the tip of the second.
- $\vec{A} + \vec{B} = \vec{C}$
- The sum of two vectors is called the resultant (often denoted by $\vec{R}$ rather than $\vec{C}$)
- Vector addition is commutative. $\vec{A} + \vec{B} = \vec{B} + \vec{A} = \vec{C}$
- One method of determining the magnitude and angle of the resultant is to use geometry, the law of cosines, and/or the law of sines.

Example: If $\vec{A} = 10.0\angle 36.9°$ and $\vec{B} = 5.00\angle 143.1°$, determine the magnitude and angle of $\vec{C} = \vec{A} + \vec{B}$.



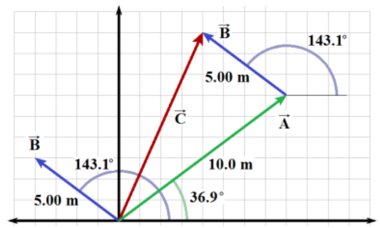
- First draw vectors $\vec{A}$ and $\vec{B}$.
- Draw a second vector $\vec{B}$ starting at the tip of vector $\vec{A}$.
- Draw in vector $\vec{C}$ from the tail of $\vec{A}$ to the tip of $\vec{B}$.

If you know the length of two sides of a triangle and the angle in between them, the law of cosines allows you to determine the length of the third side.

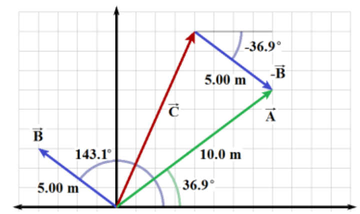
- We need to find angle γ, which is the sum of angles Φ and Ψ.
  - $\Phi = 180° - 143.1 = 36.9°$
  - $\Psi = \theta_A = 36.9°$
  - $\gamma = \Psi + \Phi = 36.9° + 36.9° = 73.8°$
- Law of Cosines: $C^2 = A^2 + B^2 - 2AB\text{Cos}(\gamma)$
  - $C^2 = (10.0\text{ m})^2 + (5.00\text{ m})^2 - 2(10.0\text{ m})(5.00\text{ m})\text{Cos}(73.8°)$
  - $C^2 = 100.\text{ m}^2 + 25.0\text{ m}^2 - (100.\text{ m}^2)\text{Cos}(73.8°)$
  - $C^2 = 100\text{ m}^2 + 25.0\text{ m}^2 - 27.9\text{ m}^2 = 97.1\text{ m}^2$
  - $C = \sqrt{97.1\text{ m}^2} = 9.85398\text{ m} = 9.85\text{ m}$
- Law of Sines: $\frac{\text{Sin}\,\alpha}{A} = \frac{\text{Sin}\,\beta}{B} = \frac{\text{Sin}\,\gamma}{C}$
  - $\text{Sin}\,\beta = \frac{B}{C}\text{Sin}\,\gamma$
  - $\beta = \text{Sin}^{-1}\left(\frac{B}{C}\text{Sin}\,\gamma\right) = \text{Sin}^{-1}\left\{\frac{5.00\text{ m}}{9.85398\text{ m}}\text{Sin}(73.8°)\right\} = 29.1608°$
  - $\theta_C = \beta + \theta_A = 29.1608° + 36.9° = 66.1°$
- $\vec{C} = 9.85\text{ m}\angle 66.1°$

Subtracting Vectors

- To subtract, multiply a vector by "-1" and add: $\vec{C} - \vec{B} = \vec{C} + (-\vec{B})$
- In the previous example, since $\vec{A} + \vec{B} = \vec{C}$, then $\vec{A} = \vec{C} - \vec{B}$

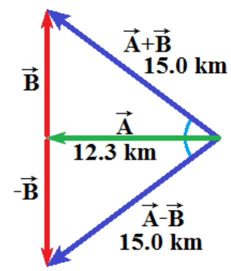


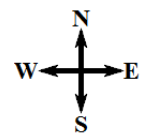
Addition $\vec{A} + \vec{B} = \vec{C}$



Subtraction $\vec{A} = \vec{C} - \vec{B} = \vec{C} + (-\vec{B})$

Example: Vector $\vec{A}$ has a magnitude of 12.3 km and points due west. Vector $\vec{B}$ points due north. If $\vec{A} + \vec{B}$ has a magnitude of 15.0 km, then a) what is the magnitude of $\vec{B}$? And b) what is the direction of $\vec{A} + \vec{B}$ relative to due west? If $\vec{A} - \vec{B}$ has a magnitude of 15.0 km, then c) what is the magnitude of $\vec{B}$? And d) what is the direction of $\vec{A} - \vec{B}$ relative to due west?





a) $\vec{C} = \vec{A} + \vec{B} \qquad A^2 + B^2 = C^2 \qquad B^2 = C^2 - A^2$

$$B = \sqrt{C^2 - A^2} = \sqrt{(15.0\text{ km})^2 - (12.3\text{ km})^2} = 8.6\text{ km}$$

b) $\theta = \text{Cos}^{-1}\left(\frac{A}{C}\right) = \text{Cos}^{-1}\left(\frac{12.3\text{ km}}{15.0\text{ km}}\right) = 34.9° \quad$ N of W

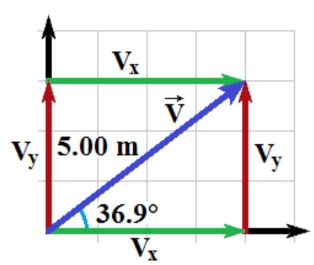
c) $\vec{C} = \vec{A} - \vec{B} \qquad A^2 + B^2 = C^2 \qquad B^2 = C^2 - A^2$

$$B = \sqrt{C^2 - A^2} = \sqrt{(15.0\text{ km})^2 - (12.3\text{ km})^2} = 8.6\text{ km}$$

d) $\theta = \text{Cos}^{-1}\left(\frac{A}{C}\right) = \text{Cos}^{-1}\left(\frac{12.3\text{ km}}{15.0\text{ km}}\right) = 34.9° \quad$ S of W

Vector Components

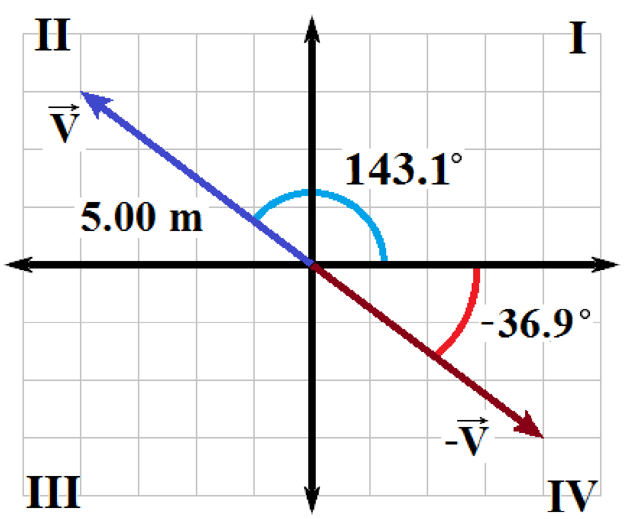
- Vectors can be broken into components which point along the axis of a reference frame (coordinate system).
- Collectively, the components are equivalent to the original vector.
- In some cases, computations are more easily done with components than the original vector.

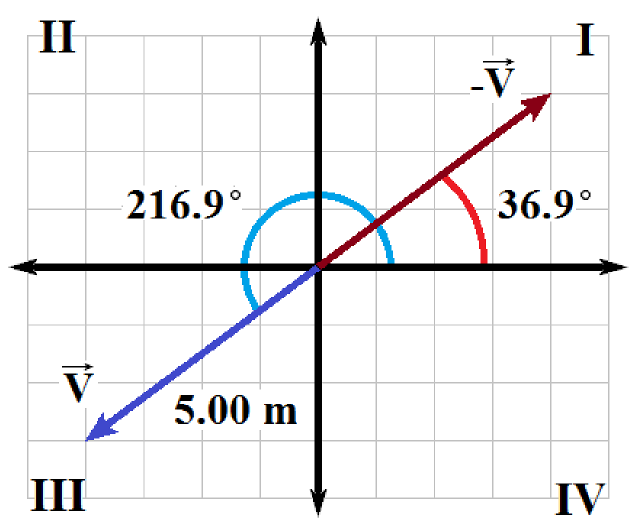


- To determine the horizontal and vertical components of a vector (V_x and V_y) use trigonometry.
  - $\text{Cos}(\theta_V) = \frac{\text{Adjacent Side}}{\text{Hypotenuse}} = \frac{V_x}{V} \qquad V_x = V \cdot \text{Cos}(\theta_V)$
  - $\text{Sin}(\theta_V) = \frac{\text{Opposite Side}}{\text{Hypotenuse}} = \frac{V_y}{V} \qquad V_y = V \cdot \text{Sin}(\theta_V)$

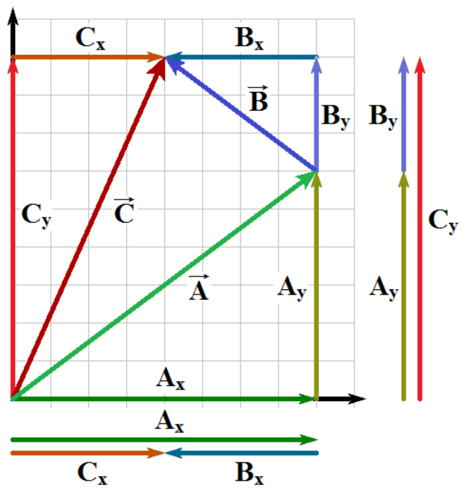
The previous equations are only valid when the angle (θ_V) is given with respect to the positive x-axis. In many cases angles are defined differently.

- For example, if $\vec{V} = (5.00\text{ m})\angle 36.9°$
  - $V_x = V \cdot \text{Cos}(\theta_V) = (5.00\text{ m})\text{Cos}(36.9°) = 4.00\text{ m}$
  - $V_y = V \cdot \text{Sin}(\theta_V) = (5.00\text{ m})\text{Sin}(36.9°) = 3.00\text{ m}$
- To determine the magnitude and angle from the horizontal and vertical components of a vector (V_x and V_y) use the Pythagorean theorem and an inverse trigonometric function.
  - $V = |\vec{V}| = \sqrt{V_x^2 + V_y^2}$
  - If $V_x > 0$, then $\theta_V = \text{Tan}^{-1}\left(\frac{V_y}{V_x}\right)$
  - If $V_x < 0$, then $\theta_V = \text{Tan}^{-1}\left(\frac{V_y}{V_x}\right) \pm 180°$
  - Calculators determine the quadrant based on the sign of the argument in Tan^{-1}, placing it in quadrant I (angles from 0 to 90°) if V_y/V_x is positive and in quadrant IV (angles from 0 to -90°) if V_y/V_x is negative.
  - This works in quadrant I (where V_x and V_y are positive and the argument, V_y/V_x is positive) and quadrant IV (where V_x is positive, V_y is negative, and the argument, V_y/V_x is negative).
  - In quadrant II (where V_x is negative, V_y is positive, and the argument, V_y/V_x is negative), it returns an answer in quadrant IV, which is the angle of $-\vec{V}$. Adding or subtracting 180° corrects this.
  - In quadrant III (where V_x and V_y are negative, V_y/V_x is positive), it returns an answer in quadrant I, which is the angle of $-\vec{V}$. Adding or subtracting 180° corrects this.





Vector Addition with Components



- Break vectors into components.
- Add corresponding components to get the resultant's components
  - $\vec{A} + \vec{B} = \vec{C}$
  - $A_x + B_x = C_x$
  - $A_y + B_y = C_y$
- Find resultant's magnitude and angle from the resultant's components.

Sometimes it's easier to add components. Sometimes it's easier to use the law of cosines. Learn both.

Example: Find $\vec{C} = \vec{A} + \vec{B}$, when $\vec{A} = (7.28\text{ m})\angle 15.95°$ and $\vec{B} = (5.00\text{ m})\angle 36.87°$.

$$C_x = A_x + B_x = A \cdot \text{Cos}(\theta_A) + B \cdot \text{Cos}(\theta_B)$$

$$C_x = (7.28\text{ m}) \cdot \text{Cos}(15.95°) + (5.00\text{ m}) \cdot \text{Cos}(36.87°) = 7.00\text{ m} + 4.00\text{ m} = 11.00\text{ m}$$

$$C_y = A_y + B_y = A \cdot \text{Sin}(\theta_A) + B \cdot \text{Sin}(\theta_B)$$

$$C_y = (7.28\text{ m}) \cdot \text{Sin}(15.95°) + (5.00\text{ m}) \cdot \text{Sin}(36.87°) = 2.00\text{ m} + 3.00\text{ m} = 5.00\text{ m}$$

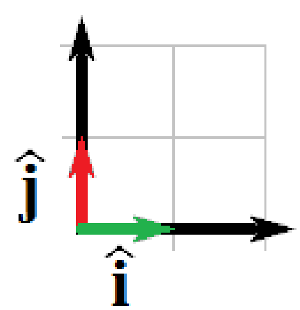
$$C = \sqrt{C_x^2 + C_y^2} = \sqrt{(11.00\,m)^2 + (5.00\,m)^2} = 12.08\,m$$

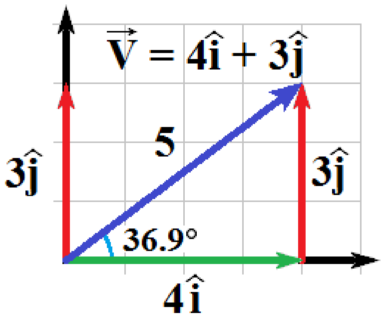
$$\theta_C = \text{Tan}^{-1}\left(\frac{C_y}{C_x}\right) = \text{Tan}^{-1}\left(\frac{5.00\text{ m}}{11.00\text{ m}}\right) = 24.44°$$

$$\vec{C} = 12.08\text{ m} \angle 24.4°$$

Unit Vectors

- Unit vectors are vectors of magnitude 1. Usually these vectors align with coordinate axes.
- Multiplying a unit vector by a scalar creates a vector with length equal to the scalar pointing along a coordinate axis.
- Any vector may be represented as the sum of it's horizontal and vertical components multiplied by their respective unit vectors. $\vec{V} = V\angle\theta_V = V_x\hat{\imath} + V_y\hat{\jmath}$





For example,

$$\vec{V} = V\angle\theta_V = (5.00\text{ m})\angle 36.9° = (4.00\text{ m})\hat{\imath} + (3.00\text{m})\hat{\jmath}$$

- This creates a convenient notation for vector addition.

$$\vec{A} + \vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath}) + (B_x\hat{\imath} + B_y\hat{\jmath}) = (A_x\hat{\imath} + B_x\hat{\imath}) + (A_y\hat{\jmath} + B_y\hat{\jmath})$$

$$\vec{A} + \vec{B} = (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} = C_x\hat{\imath} + C_y\hat{\jmath} = \vec{C}$$

Example: Find the resultant of 3 displacement vectors by means of the component method. The three vectors are $\vec{A} = (5.00\text{ m})\angle 160°$, $\vec{B} = (5.00\text{ m})\angle 60°$, and $\vec{C} = (4.00\text{ m})\angle 270°$.

$$\vec{A} = (5.00\text{ m})\angle 160° = (5.00\text{ m})\text{Cos}(160°)\hat{\imath} + (5.00\text{ m})\text{Sin}(160°)\hat{\jmath} = (-4.70\text{ m})\,\hat{\imath} + (1.71\text{ m})\,\hat{\jmath}$$

$$\vec{B} = (5.00\text{ m})\angle 60° = (5.00\text{ m})\text{Cos}(60°)\hat{\imath} + (5.00\text{ m})\text{Sin}(60°)\hat{\jmath} = (2.50\text{ m})\hat{\imath} + (4.33\text{ m})\hat{\jmath}$$

$$\vec{C} = (4.00\text{ m})\angle 270° = (4.00\text{ m})\text{Cos}(270°)\hat{\imath} + (4.00\text{ m})\text{Sin}(270°)\hat{\jmath} = (-4.00\text{m})\hat{\jmath}$$

$$\vec{R} = \vec{A} + \vec{B} + \vec{C} = \underbrace{\{(-4.70\text{ m})\hat{\imath} + (1.71\text{ m})\hat{\jmath}\}}_{\vec{A}} + \underbrace{\{(2.50\text{ m})\hat{\imath} + (4.33\text{ m})\hat{\jmath}\}}_{\vec{B}} + \underbrace{\{(-4.00\text{m})\hat{\jmath}\}}_{\vec{C}}$$

$$\vec{R} = \{(-4.70\text{ m}) + (2.50\text{ m})\}\hat{\imath} + \{(1.71\text{ m}) + (4.33\text{ m}) + (-4.00\text{m})\}\hat{\jmath}$$

$$\vec{R} = (-2.20\text{ m})\hat{\imath} + (2.04\text{ m})\hat{\jmath}$$

$$R = |\vec{R}| = \sqrt{R_x^2 + R_y^2} = \sqrt{(2.20\text{ m})^2 + (2.04\text{ m})^2} = 3.00\text{ m}$$

$$\theta_R = \text{Tan}^{-1}\left(\frac{R_y}{R_x}\right) + 180° = \text{Tan}^{-1}\left(\frac{2.04\text{ m}}{-2.20\text{ m}}\right) + 180° = -42.8° + 180° = 137.2°$$

$$\vec{R} = (-2.20\text{ m})\hat{\imath} + (2.04\text{ m})\hat{\jmath} = (3.00\text{ m})\angle 137.2°$$