Lecture 27: Fluids and Simple Harmonic Motion

Physics for Engineers & Scientists (Giancoli): Chapters 13 & 14
University Physics V1 (Openstax): Chapters 14 & 15

Example: During a tornado the winds and pressure can be sufficient to rip the roofs off houses. The roof of a cabin is 10.0 m by 7.00 m. The air outside is moving at 67.0 m/s (roughly 150 mph). You may assume that the difference in height from the top of the roof to the bottom is negligible. The density of air is 1.225 kg/m^3. How much force is exerted on the roof?

$$F_{Net} = F_{inside} - F_{outside} = P_{inside}A - P_{outside}A = (P_{inside} - P_{outside})A = (P_1 - P_2)LW$$

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

As stated in the problem, $y_1 \approx y_2$ (ρgy terms cancel). Also, the air is still inside ($v_1 = 0$).

$$P_1 = P_2 + \frac{1}{2}\rho v_2^2$$

$$P_1 - P_2 = \frac{1}{2}\rho v_2^2 = \frac{1}{2}\left(1.225\frac{kg}{m^3}\right)\left(67.0\frac{m}{s}\right)^2 = 2{,}749.5\ Pa$$

$$F_{Net} = (P_1 - P_2)A = (2{,}749.5\ Pa)(10.0\ m)(7.00m) = 192\ kN$$

192 kN is roughly equal to 43,000 lbs.

Example: The top was removed from an old silo to convert it to collecting and storing rain water. The silo is 10.0 m high, 2.00 m in radius, and full to the open top with water. A poorly placed shot from a rifle puts a hole in the side of the silo that is 1.00 cm in diameter and 1.60 m above the ground. Determine the velocity of the water as it streams out the hole.

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

P_1 and P_2 are both 1 atm (open to the air) and will cancel.

$$\tfrac{1}{2}\rho v_1^2 + \rho g y_1 = \tfrac{1}{2}\rho v_2^2 + \rho g y_2 \qquad \rho g y_1 - \rho g y_2 = \tfrac{1}{2}\rho v_2^2 - \tfrac{1}{2}\rho v_1^2 \qquad \rho g(y_1 - y_2) = \tfrac{1}{2}\rho(v_2^2 - v_1^2)$$

Need to get v_1 (top of the silo) in terms of v_2 (coming out the bullet hole).

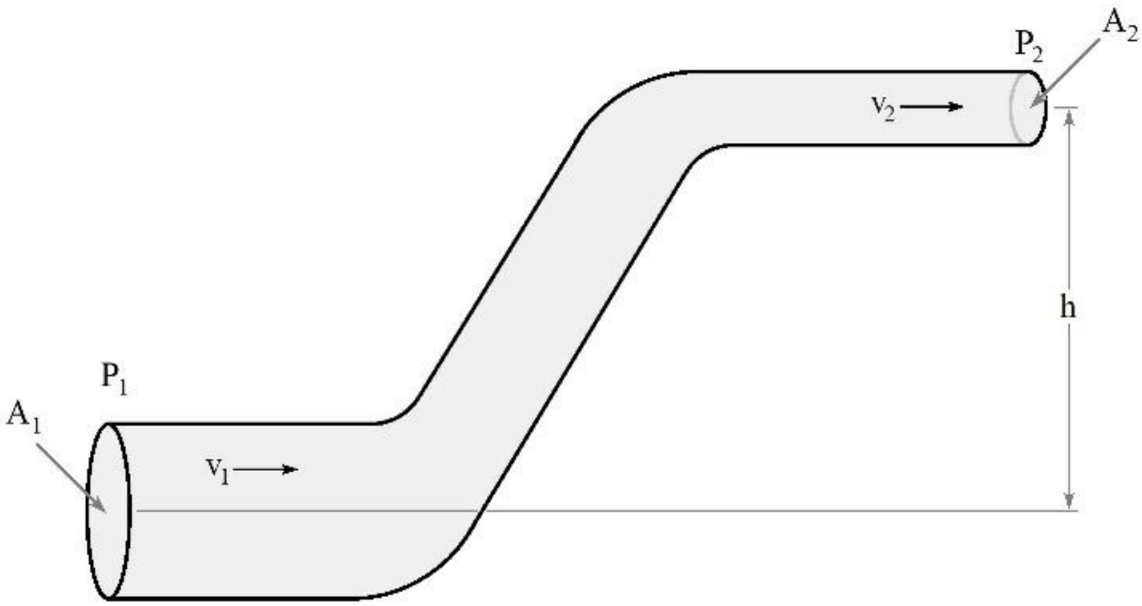
$$A_1 v_1 = A_2 v_2 \qquad v_1 = \frac{A_2}{A_1}v_2 = \frac{\pi r_2^2}{\pi r_1^2}v_2 = \left(\frac{r_2}{r_1}\right)^2 v_2 = \left(\frac{0.005\ m}{2.00\ m}\right)^2 v_2 = (6.25 \times 10^{-6})\ v_2$$

As v_1 is roughly a million times smaller than v_2, the v_1^2 is negligible compared to v_2^2. Keep this in mind. In many cases, one of the KE terms is negligible.

$$\rho g(y_1 - y_2) = \tfrac{1}{2}\rho v_2^2 \qquad v_2^2 = 2g(y_1 - y_2)$$

$$v_2 = \sqrt{2g(y_1 - y_2)} = \sqrt{2\left(9.80\frac{m}{s^2}\right)(10.0\ m - 1.60\ m)} = 12.8\frac{m}{s}$$

Example: In order to clear an obstacle, an oil pipeline rises h = 12.5 m in height. As it does the pipeline narrows from a diameter of 1.20 m to a diameter of 0.625 m. The density of the crude oil is 827 kg/m^3, and the velocity of the oil at the bottom is v_1 = 1.25 m/s. If the pressure at the top (P_2), can't exceed 56.7 atm, determine the maximum pressure of P_1.



$$A_1 v_1 = A_2 v_2 \qquad v_2 = \frac{A_1}{A_2} v_1 = \frac{\pi r_1^2}{\pi r_2^2} v_1 = \left(\frac{r_1}{r_2}\right)^2 v_1 = \left(\frac{d_1}{d_2}\right)^2 v_1 = \left(\frac{1.20\ m}{0.625\ m}\right)^2 \left(1.25 \frac{m}{s}\right) = 4.608 \frac{m}{s}$$

$$P_2 = 56.7\ atm \left(\frac{1.013 \times 10^5\ Pa}{1\ atm}\right) = 5.74371 \times 10^6\ Pa$$

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

$$P_1 = P_2 + \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g (y_2 - y_1) = P_2 + \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g h$$

$$P_1 = 5.74371 \times 10^6\ Pa + \frac{1}{2}\left(827 \frac{\mathrm{kg}}{\mathrm{m}^3}\right)\left[\left(4.608 \frac{m}{s}\right)^2 - \left(1.25 \frac{m}{s}\right)^2\right] + \left(827 \frac{\mathrm{kg}}{\mathrm{m}^3}\right)\left(9.80 \frac{\mathrm{m}}{\mathrm{s}^2}\right)(12.5\mathrm{m})$$

$$P_1 = 5.74371 \times 10^6\ Pa + 8134\ Pa + 101{,}308\ Pa = 5.85 \times 10^6\ Pa$$

Simple Harmonic Motion

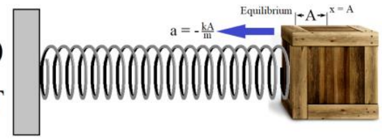
- Simple Harmonic Motion is a repetitive (periodic) state of motion that occurs when the magnitude of the restoring force is proportional to the displacement from equilibrium.

One place this type of behavior occurs is when a mass is attached to a spring and allowed to slide across a frictionless surface.

- If we pull the mass a distance A away from equilibrium, the Restoring Force of the spring will pull it back towards the equilibrium position.

$t = 0$

$t = nT$



$x = x_{max} = A$
$v = 0$
$a = -a_{max} = -\frac{kA}{m} = -\omega^2 A$

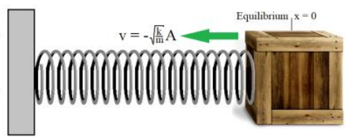
- The <u>Amplitude</u> (A) is the maximum displacement of the system from equilibrium.
- The mass is released from rest in this case. $v_0 = 0$.
- The acceleration is: $a = \frac{F}{m} = \frac{-kx}{m} = \frac{-kA}{m}$
- As will be shown later, the angular frequency (ω) of the system is: $\omega = \sqrt{\frac{k}{m}}$
- The object will return to this exact state of the beginning of every period

$$t = nT, \quad for\ n = 0, 1, 2, \ldots$$

- After a quarter of the period, the mass will have returned to its equilibrium position.

$t = \frac{1}{4}T$

$t = (n + \frac{1}{4})T$



$x = 0$
$v = -v_{max} = -\sqrt{\frac{k}{m}}A = -\omega A$
$a = 0$

- At equilibrium (x = 0) the net force is zero. This means there is no longer any acceleration.
- One it passes equilibrium it will begin to decelerate, making this the maximum speed.
- We can calculate that speed using conservation of energy:

$$E_{Init} = U_{Elastic} = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \qquad E_{Final} = KE = \frac{1}{2}mv^2$$

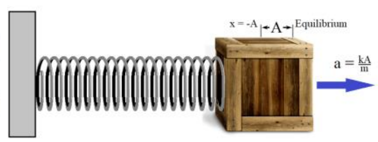
$$E_{Init} = E_{Final} \qquad \frac{1}{2}kA^2 = \frac{1}{2}mv^2 \qquad kA^2 = mv^2 \qquad \frac{k}{m}A^2 = v^2 \qquad v = \sqrt{\frac{k}{m}}A = \omega A$$

- The object will return to this state every period: $t = (n + \frac{1}{4})T, \quad for\ n = 0, 1, 2, \ldots$

- After half of the period, the mass will have come to rest again.

$t = \frac{1}{2}T$

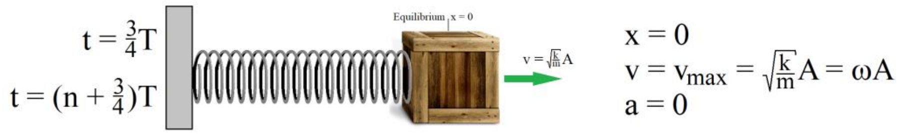
$t = (n + \frac{1}{2})T$



$x = -x_{max} = -A$
$v = 0$
$a = a_{max} = \frac{kA}{m} = \omega^2 A$

- The amplitude has maximum magnitude again, this time on the negative side: x = -A.
- The mass has come to rest again. $v_0 = 0$.
- The acceleration is: $a = \frac{F}{m} = \frac{-kx}{m} = \frac{kA}{m}$
- The object will return to this state every period: $t = (n + \frac{1}{2})T, \quad for\ n = 0, 1, 2, \ldots$

- After three quarters of the period, the mass will have returned to its equilibrium position again.

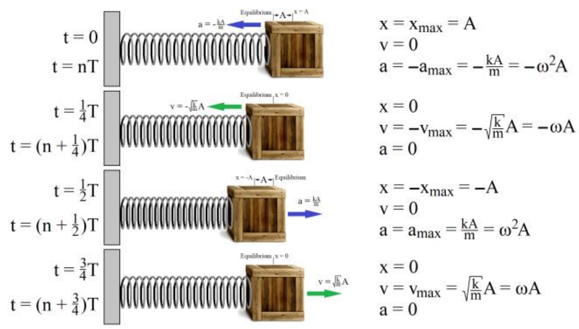


- At equilibrium (x = 0) the net force is zero. This means there is no longer any acceleration.
- One it passes equilibrium it will begin to decelerate, making this the maximum speed.

$$v = \sqrt{\frac{k}{m}}A = \omega A$$

- The object will return to this state every period: $t = (n + \frac{3}{4})T, \quad for\ n = 0, 1, 2, \ldots$

- After one full period, the object returns to its initial position and state and the cycle begins again.



- The energy of the system continually changes from potential energy to kinetic energy and back.
- The velocity at any position can be calculated using conservation of energy:

$$E_{system} = E_{Init} = \frac{1}{2}kA^2 \qquad E_{Final} = U_{Elastic} + KE = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

$$E_{Init} = E_{Final} \qquad \frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \qquad \frac{1}{2}kA^2 - \frac{1}{2}kx^2 = \frac{1}{2}mv^2$$

$$\frac{1}{2}k(A^2 - x^2) = \frac{1}{2}mv^2 \qquad k(A^2 - x^2) = mv^2 \qquad \frac{k}{m}(A^2 - x^2) = v^2 \qquad v = \sqrt{\frac{k}{m}(A^2 - x^2)}$$

- General Solution

$$F = ma \qquad -kx = m\frac{d^2x}{dt^2} \qquad m\frac{d^2x}{dt^2} + kx = 0 \qquad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

The solution to this well-known differential equation is of the form: $x(t) = C_1 \cos \omega t + C_2 \sin \omega t$

The given initial position at t=0 is x=A. $x(0) = C_1 \cos 0° + C_2 \sin 0° = C_1 = A$

The given initial position is a maximum. $C_2 = 0$

$$x(t) = A \cos \omega t \qquad \frac{d^2x}{dt^2} = \frac{d^2}{dt^2}(A \cos \omega t) = \frac{d}{dt}(-\omega A \sin \omega t) = -\omega^2 A \cos \omega t$$

Plug into the initial different equation:

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad -\omega^2 A\cos\omega t + \frac{k}{m}A\cos\omega t = 0 \qquad \left(\frac{k}{m} - \omega^2\right)A\cos\omega t = 0$$

As A ≠ 0 and cos(ωt) ≠ 0 (at least it isn't for all values of t):

$$\frac{k}{m} - \omega^2 = 0 \qquad \omega = \sqrt{\frac{k}{m}} \qquad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Which gives us our solution: $x(t) = A\cos\omega t$ *with* $\omega = \sqrt{\frac{k}{m}}$

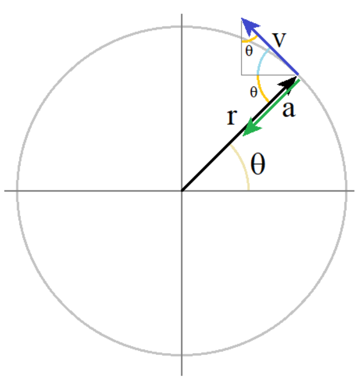
$$v = \frac{dx}{dt} = \frac{d}{dt}(A\cos\omega t) = -\omega A\sin\omega t \qquad v_{max} = \omega A = \sqrt{\frac{k}{m}}A$$

$$a = \frac{d^2x}{dt^2} = \frac{d^2}{dt^2}(A\cos\omega t) = -\omega^2 A\cos\omega t \qquad a_{max} = \omega^2 A = \frac{k}{m}A$$

This solution is only valid when the initial position is a maximum.

Simple Harmonic Motion vs. Uniform Circular Motion

- The x- and y-components of uniform circular motion are in simple harmonic motion.
  - Simple harmonic motion can be viewed as a 1-dimensional view of uniform circular motion.
  - Uniform circular motion can be viewed as 2-dimensional simple harmonic motion where the two components are 90° out of phase.



Let the radius of the circle be the amplitude (A). $r = A$

There is no angular acceleration. $\theta = \omega t + \theta_0$

$$x = r\cos\theta = A\cos(\omega t + \theta_0)$$

$$v_x = -v\sin\theta = -\omega r\sin(\omega t + \theta_0) = -\omega A\sin(\omega t + \theta_0)$$

$$a_x = -a\cos\theta = -\frac{v^2}{r}\cos(\omega t + \theta_0) = -\frac{(\omega r)^2}{r}\cos(\omega t + \theta_0)$$

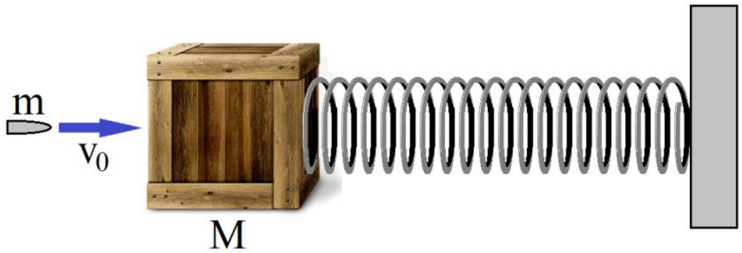
$$a_x = -\omega^2 r\cos(\omega t + \theta_0) = -\omega^2 A\cos(\omega t + \theta_0)$$

Example: Military aircraft and pilots are tested to ensure they can withstand accelerations of 9g (88.2 m/s^2). To ensure that communication equipment can withstand these g-forces it is placed on an oscillating table that shifts back and forth in simple harmonic motion at a frequency of 5.25 Hz. To ensure that the equipment is tested at a maximum acceleration of 9g, what amplitude is needed?

$$a_{max} = \omega^2 A = (2\pi f)^2 A = 4\pi^2 f^2 A = 9g$$

$$A = \frac{9g}{4\pi^2 f^2} = \frac{9\left(9.80\,\frac{m}{s^2}\right)}{4\pi^2(5.25\ Hz)^2} = 25.5\ cm$$

Example: A 50.0 kg block is attached to a spring (k = 450 N/m), which in turn is attached to a wall. The block is at rest when it is struck by a bullet with a trajectory that would pass straight down the center of the spring. The bullet becomes lodged in the block, and sends it into simple harmonic motion with frequency of 0.4765 Hz and amplitude 50.5 cm. Determine the mass and the initial velocity of the bullet.



$$\omega = \sqrt{\frac{k}{m+M}} = 2\pi f \qquad \frac{k}{m+M} = 4\pi^2 f^2 \qquad m + M = \frac{k}{4\pi^2 f^2}$$

$$m = \frac{k}{4\pi^2 f^2} - M = \frac{\left(450\,\frac{N}{m}\right)}{4\pi^2 (0.4765)^2} - 50.0\,kg = 0.20269\,kg$$

Conservation of momentum relates the bullet velocity (v_0) to V.

$$mv_0 = (m+M)V \qquad v_0 = \left(1 + \frac{M}{m}\right)V$$

Conservation of energy (after the collision) relates V to A.

Or…

The velocity of the block and bullet (V) right after the collision occurs at equilibrium.
That means V is the maximum velocity.

$$V = v_{max} = \omega A = 2\pi f A$$

$$v_0 = \left(1 + \frac{M}{m}\right)V = \left(1 + \frac{M}{m}\right)2\pi f A = \left(1 + \frac{50.0\,kg}{0.20269\,kg}\right)2\pi(0.4765\text{ Hz})(0.505\,m) = 374\,\frac{m}{s}$$