

Lecture 21: Rotational Dynamics and Moment of Inertia

Physics for Engineers & Scientists (Giancoli): Chapter 10

University Physics V1 (Openstax): Chapter 10

Rotational Dynamics

- As rotational kinematics showed many similarities with one-dimensional translational kinematics, we can expect more similarities to appear in dynamics, but there are some differences too.

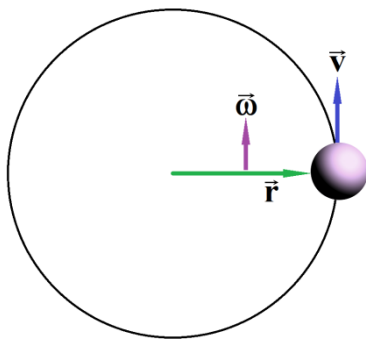
Concept	Translational (1D)	Rotational	Relationship
Position	x (or S)	θ	$S = r\theta$
Velocity	v	ω	$v = r\omega$
Acceleration	a	α	$a_T = r\alpha$ $a_R = \omega^2 r$
Cause of Acceleration	F	τ (Torque)	$\vec{\tau} = \vec{r} \times \vec{F}$ ($ \tau = rF \sin \theta$)
Inertia	m	I (Moment of Inertia)	$dI = r^2 dm$
Newton's 2 nd Law	$\vec{F} = m\vec{a}$	$\vec{\tau} = I\vec{\alpha}$	
Work	$W = Fd$	$W = \tau\theta$	
Kinetic Energy	$KE_{\text{Translational}} = \frac{1}{2}mv^2$	$KE_{\text{Rotational}} = \frac{1}{2}I\omega^2$	
Momentum	$\vec{P} = m\vec{v}$	$\vec{L} = I\vec{\omega}$	$\vec{L} = \vec{r} \times \vec{P}$
Force/Momentum	$\vec{F} = \frac{d\vec{P}}{dt}$	$\vec{\tau} = \frac{d\vec{L}}{dt}$	

Moment of Inertia (I)

- The moment of inertia is the rotational equivalent of mass (a resistance to being spun).

An object in uniform circular motion can be looked at as either rotational motion or as in linear translational motion (at least temporarily).

The kinetic energy of this motion should be the same either way it is calculated. This allows us to determine a relationship between moment of Inertia (I) and mass (m).



$$KE_{\text{Translational}} = \frac{1}{2}mv^2 = \frac{1}{2}m(r\omega)^2 = \frac{1}{2}mr^2\omega^2$$

$$KE_{\text{Rotational}} = \frac{1}{2}I\omega^2 \quad \frac{1}{2}mr^2\omega^2 = \frac{1}{2}I\omega^2$$

$$I = mr^2$$

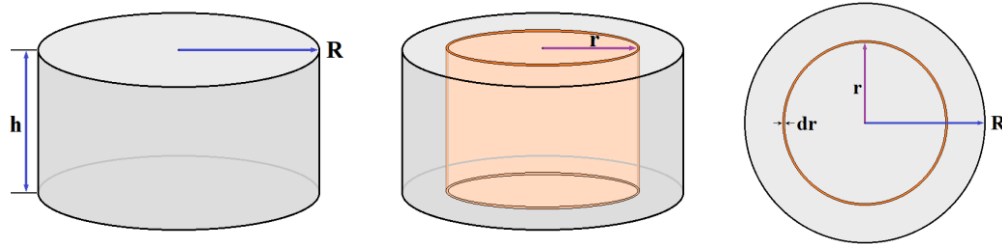
Note: This is only valid when all the mass is located at the same distance (r) from the axis of rotation.

- When the mass is not all at the same value of r, you must calculate the mass at each value of r and sum them all up. As there is often a continuous distribution of r values, summing may mean integration.

$$dI = r^2 dm \quad I = \int_{r_1}^{r_2} r^2 dm \quad dm = \rho dV$$

Example: Determine the moment of inertia of a solid cylindrical disk of uniform density, mass M , and radius R .

Step 1: Split mass into small pieces, each with the same value of r (as r appears in our equation). In this case, it will be infinitesimally thin cylindrical shells.



The left image shows the cylinder and its dimensions (R and h).

The center image shows one of the small pieces, a hollow cylinder in the center (orange).

The right image is a view from above. The thickness of the shell is dr .

Step 2: Find ' dm ', the mass of a representative shell.

$$dm = \rho dV = \rho A dr = \rho(2\pi r h) dr = (2\pi \rho h) r dr \quad \rho = \frac{M}{V} = \frac{M}{\pi R^2 h}$$

A is the surface area of the cylindrical shell. The length is the circumference and the width is h .

$$dm = (2\pi \rho h) r dr = \left(2\pi \frac{M}{\pi R^2 h} h\right) r dr = \frac{2M}{R^2} r dr$$

Be careful with your variables. Don't confuse R and r . R is the radius of the cylindrical disk. r is the radius of the thin cylindrical shell.

Step 3: Plug in and integrate.

$$I = \int_0^R r^2 dm = \int_0^R r^2 \left(\frac{2M}{R^2} r dr\right) = \frac{2M}{R^2} \int_0^R r^3 dr = \frac{2M}{R^2} \left[\frac{r^4}{4}\right]_0^R = \frac{2M}{R^2} \left[\frac{R^4}{4}\right] = \frac{1}{2} MR^2$$

Example: Find the moment of inertia for a flat (right) triangular plate if it is rotating around the y -axis. You may assume uniform thickness and density.

Step 1: Split mass into small pieces, each with the same value of r (as r appears in our equation).

In this case the distance from the axis of rotation (the y -axis), is just x . ($r = x$)

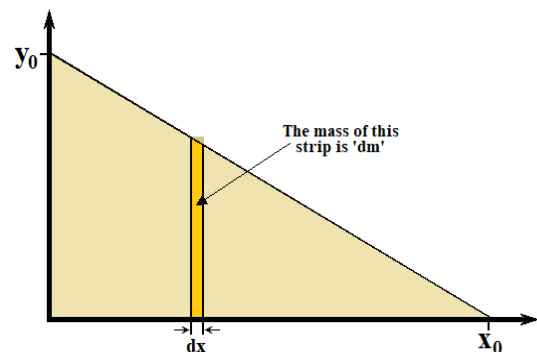
Step 2: Find ' dm ', the mass of the strip.

$$dm = \rho dV = \rho z_0 A = \rho z_0 y dx$$

In this instance, we must find y as a function of x (since y varies with x).

$$y = mx + b \quad m = \frac{\Delta y}{\Delta x} = \frac{0 - y_0}{x_0 - 0} = -\frac{y_0}{x_0}$$

$$b = y_0 \quad dm = \rho z_0 y dx = \rho z_0 (mx + b) dx$$



We also need the value of ρ . $\rho = \frac{M}{V} = \frac{M}{At} = \frac{M}{\frac{1}{2}bht} = \frac{2M}{bht} = \frac{2M}{x_0 y_0 z_0}$

Step 3: Plug in and integrate.

$$I = \int_0^{x_0} r^2 dm = \int_0^{x_0} x^2 dm = \int_0^{x_0} x^2 \rho z_0 (mx + b) dx = \int_0^{x_0} x^2 \frac{2M}{x_0 y_0 z_0} z_0 (mx + b) dx$$

$$I = \frac{2M}{x_0 y_0} \int_0^{x_0} (mx^3 + bx^2) dx = \frac{2M}{x_0 y_0} \left[\frac{1}{4} mx^4 + \frac{1}{3} bx^3 \right]_0^{x_0} = \frac{2M}{x_0 y_0} \left[\frac{1}{4} mx_0^4 + \frac{1}{3} bx_0^3 \right]$$

$$I = \frac{2M}{x_0 y_0} \left[\frac{1}{4} \left(-\frac{y_0}{x_0} \right) x_0^4 + \frac{1}{3} (y_0) x_0^3 \right] = \frac{2M}{x_0 y_0} \left[-\frac{1}{4} y_0 x_0^3 + \frac{1}{3} y_0 x_0^3 \right] = \frac{2M}{x_0 y_0} \left[\frac{1}{12} y_0 x_0^3 \right] = \frac{1}{6} M x_0^2$$