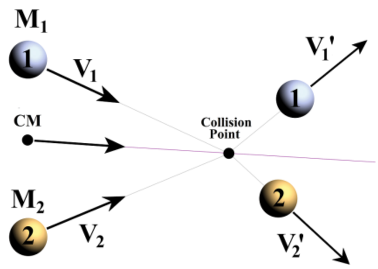
Lecture 20: Center of Mass, and Rotational Kinematics

Physics for Engineers & Scientists (Giancoli): Chapters 9 & 10
University Physics V1 (Openstax): Chapters 9 & 10

- The center of mass of a system of particles of net mass M, moves like a particle of net mass M. When subject to the next external forces it accelerates according to $F_{Net} = Ma$.



$$v_{cm} = \frac{dx_{cm}}{dt} = \frac{d}{dt}\left(\frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}\right)$$

$$v_{cm} = \frac{d}{dt}\left(\frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{M}\right)$$

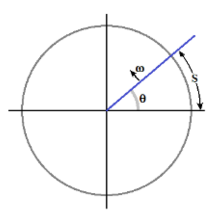
$$v_{cm} = \frac{m_1}{M}\frac{dx_1}{dt} + \frac{m_2}{M}\frac{dx_2}{dt} + \frac{m_3}{M}\frac{dx_3}{dt} + \cdots$$

$$v_{cm} = \frac{m_1v_1}{M} + \frac{m_2v_2}{M} + \frac{m_3v_3}{M} + \cdots = \frac{m_1v_1'}{M} + \frac{m_2v_2'}{M} + \frac{m_3v_3'}{M} + \cdots$$

Rotational Kinematics

For rotating objects, velocity is not a universal variable. Different parts of the object move at different speeds. Distance (x) is also not a universal variable. Consequently, these are not good variables to describe rotational motion.

- Angular Position (θ) fills the role of x.
- Initial Angular Position (θ_0) is the angular position at t=0, and it fills the role of x_0.
- Angular Displacement ($\Delta\theta = \theta - \theta_0$) fills the role of Δx.
- Arc Length (S) is the distance a part of the rigid object moves.



1 rotation/revolution = 2π radians = 360°

$$S = r \cdot \Delta\theta \ (radians)$$

$$\theta = \frac{S}{r} \qquad \frac{meters}{meters} = no\ units$$

Note: 'Radians' is a 'dummy' unit.

- Angular Velocity (ω) fills the role of v. (*ω is also called 'Angular Frequency'*)
- Initial Angular Velocity (ω_0) is the angular velocity at t=0, and it fills the role of v_0.

Note: 'ω' is a lower-cased Greek letter omega. Not W.

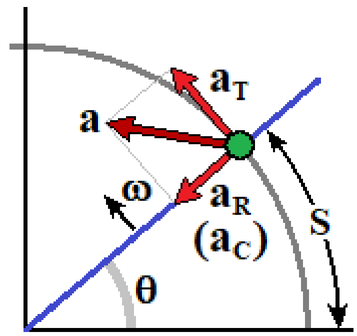
$$\omega_{Avg} = \frac{\Delta\theta}{\Delta t} = \frac{\theta - \theta_0}{t} \qquad \omega = \frac{d\theta}{dt} \qquad \text{Units: } \frac{rad}{s} \qquad r\omega = r\frac{d\theta}{dt} = \frac{d(r\theta)}{dt} = \frac{dS}{dt} = v$$

- The Period (T) is the time needed to make one full revolution.
- The Frequency (f) is the rotation rate (number of revolutions per unit time)

$$\text{Units: } 1\ \frac{\text{Revolution}}{\text{Second}} = 1\ \text{s}^{-1} = 1\ \text{Hz} \qquad 60\ \text{RPM} = 60\ \frac{\text{Revolution}}{\text{Minute}} = 1\ \text{Hz}$$

$$f = \frac{1}{T} \qquad \omega = 2\pi f = \frac{2\pi}{T} \qquad v = \frac{2\pi r}{T} = \omega r$$

- In rotational motion, Acceleration (a) gets broken down into two components.



- Tangential Accleration (a_T) changes the speed (magnitude of velocity) of an object moving in a circle.
- Radial Accleration (a_R), equivalent to centripetal acceleration (a_C), changes the direction but not the speed of an object moving in a circle.

$$a = \sqrt{a_T^2 + a_R^2}$$

- Angular Acceleration (α) fills the role of a.

$$\alpha_{Avg} = \frac{\Delta\omega}{\Delta t} = \frac{\omega - \omega_0}{t} \qquad \alpha = \frac{d\omega}{dt} \qquad \text{Units: } \frac{rad}{s^2}$$

$$r\alpha = r\frac{d\omega}{dt} = \frac{d(r\omega)}{dt} = \frac{dv}{dt} = a_T \qquad a_R = a_C = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2$$

- For the special case of constant angular acceleration, a set of equations can be found from the one-dimensional kinematic equations for constant acceleration.

$$v = v_0 + a_T t \qquad \omega r = \omega_0 r + \alpha r t \qquad \omega = \omega_0 + \alpha t$$

$$s = s_0 + \frac{1}{2}(v + v_0)t \qquad r\theta = r\theta_0 + \frac{1}{2}(r\omega + r\omega_0)t \qquad \theta = \theta_0 + \frac{1}{2}(\omega + \omega_0)t$$

$$s = s_0 + v_0 t + \frac{1}{2}a_T t^2 \qquad r\theta = r\theta_0 + r\omega_0 t + \frac{1}{2}r\alpha t^2 \qquad \theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$v^2 = v_0^2 + 2a_T(s - s_0) \qquad r^2\omega^2 = r^2\omega_0^2 + 2r\alpha(r\theta - r\theta_0) \qquad \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

  - Rotational kinematics are similar to one-dimensional kinematics (just a change of variable) and solved the same way.
  - There are four variables (θ, ω, α, and t) and two constants (θ_0 and ω_0).
  - Three of the variables are related in each of the four equations. In many cases you can find the equation you need by determining which variable is absent.

Example: A wind turbine is activated as the winds reach a threshold. The blades start from rest and accelerate uniformly to an angular velocity of 9.87 rpms in 27.4 s. Determine the angular acceleration of the blades.

Extract Data: $\omega_0 = 0$ ω = 9.87 rpms = 1.03358 rad/s t = 27.4 s α = ???

$$\omega = 9.87\left(\frac{Rev}{Min}\right)\left(\frac{1\ Min}{60\ s}\right)\left(\frac{2\pi\ rad}{1\ Rev}\right) = 1.03358\frac{rad}{s}$$

Be warned: 2π/60 = 0.10472. If you fail to do this conversion, you will be off by a factor that is close to a power of 10.

No information about position is given. The equation without position is…

$$\omega = \omega_0 + \alpha t = \alpha t \qquad \alpha = \frac{\omega}{t} = \frac{1.03358\frac{rad}{s}}{27.4\ s} = 0.03772\frac{rad}{s^2}$$

Example: A grinding wheel undergoes uniform angular acceleration from rest to 680 rad/s over 1.30 seconds. Then the power is removed and friction causes it to decelerate back to rest in 18.7 seconds. Through what angle does the wheel turn during this time?

There are two different accelerations (both constants). This requires two sets of equations, one for the acceleration and one for the deceleration. This is an odd case where both are the same.

Accelerating:

Extract Data: $\theta_0 = 0$ $\theta = ???$ $\omega_0 = 0$ $\omega = 680\frac{rad}{s}$ $\alpha =$ $t = 1.30\,s$

Equation with no α: $\theta = \theta_0 + \frac{1}{2}(\omega + \omega_0)t$ $\theta = \frac{1}{2}\omega t = \frac{1}{2}\left(680\frac{rad}{s}\right)(1.30\,s) = 442\,rad$

Decelerating:

Extract Data: $\theta_0 = 442\,rad$ $\theta = ???$ $\omega_0 = 680\frac{rad}{s}$ $\omega = 0$ $\alpha =$ $t = 18.7\,s$

Equation with no α: $\theta = \theta_0 + \frac{1}{2}(\omega + \omega_0)t$

$$\theta = \theta_0 + \frac{1}{2}\omega_0 t = 442\,rad + \frac{1}{2}\left(680\frac{rad}{s}\right)(18.7\,s) = 6800\,rad$$

Alternatively, one could note that the average angular velocity is the same value whether accelerating or decelerating.

The solution would just be $\omega_{avg}t_{net}$

$\omega_{avg} = \frac{1}{2}\omega = \frac{1}{2}\left(680\frac{rad}{s}\right) = 340\frac{rad}{s}$ $\theta = \omega_{avg}t = \left(340\frac{rad}{s}\right)(1.30\,s + 18.7\,s) = 6800\,rad$

Rotational Vectors

- While we are able to treat rotational variables one-dimensionally in most cases, they are still vectors with magnitude and direction.
- The direction of rotational vectors is defined to be either parallel or anti-parallel to the axis of rotation. Anti-parallel means parallel but pointing the opposite direction.
- If the object is rotating counter-clockwise in an xy-plane when viewed from above, then ω points upward in the z-direction.
- If the object is rotating clockwise in an xy-plane when viewed from above, then ω points downward in the negative z-direction.