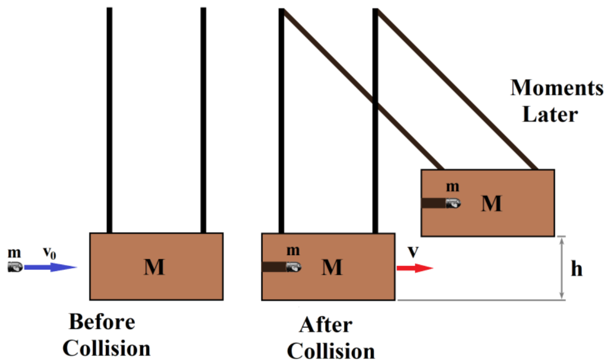
Lecture 19: Momentum in 2D and Center of Mass

Physics for Engineers & Scientists (Giancoli): Chapter 9
University Physics V1 (Openstax): Chapter 9

Ballistic Pendulum



- The ballistic pendulum is used to measure the velocity of projectiles (such as bullet).
- First, the projectile makes a completely inelastic collision with the much heavier hanging mass of a pendulum bob.
- The velocity of the pair after the collision causes the pendulum bob to swing upwards, and the height is measured. From this height we can produce the velocity of the projectile.

- To start we need to relate the 'Moments Later' image to the 'After Collision' image.
    - Can we use conservation of momentum? No, an external force (gravity) acts on the system. Momentum is not conserved.
    - There is no collision in this interval. Energy is conserved.

$$E_{Init} = E_{Final} \qquad \frac{1}{2}(m+M)v^2 = (m+M)gh \qquad v^2 = 2gh \qquad v = \sqrt{2gh}$$

- Next we need to relate the 'After Collision' image to the 'Before Collision' image.
    - Can we use conservation of energy? No, a collision occurs. Energy is not conserved.
    - There are no (horizontal) external forces in this interval. Momentum is conserved.

$$P_{Init} = P_{Final} \qquad mv_0 = (m+M)v \qquad v_0 = \left(1+\frac{M}{m}\right)v = \left(1+\frac{M}{m}\right)\sqrt{2gh}$$

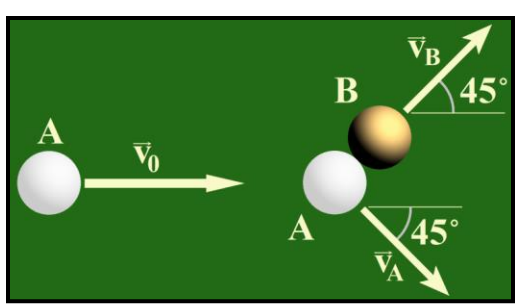
Collisions in Two Dimensions

- Conservation of momentum is applied by components for each axis of motion.

$$\vec{P}_{final} = \vec{P}_{init} \quad \text{implies} \quad P_{final-x} = P_{init-x} \quad \text{and} \quad P_{final-y} = P_{init-y}$$

- In two dimensions, conservation of motion gives two equations, allowing you to find two unknowns.

Example: The cue ball approaches a stationary ball of equal mass at 3.00 m/s. After the collision the balls separate, the velocity of each ball making a 45.0° angle with the cue ball's original path just on opposite sides. Determine the velocity of both balls after the collision.



<u>Y-Components</u>: $P_{final-y} = P_{init-y}$ $\quad m_B v_B \sin\theta_B - m_A v_A \sin\theta_A = 0$

$$m_B v_B \sin\theta_B = m_A v_A \sin\theta_A \qquad v_B = v_A$$

<u>Y-Components</u>: $P_{final-x} = P_{init-x}$ $\quad m_B v_B \cos\theta_B + m_A v_A \cos\theta_A = m_A v_0$

$$v_B \cos 45° + v_A \cos 45° = v_0 \qquad v_A \cos 45° + v_A \cos 45° = v_0$$

$$2v_A \cos 45° = v_0 \qquad v_A = \frac{v_0}{2\cos 45°} = \frac{3.00\frac{m}{s}}{2\cos 45°} = 2.12132\frac{m}{s} \qquad v_B = v_A = 2.12\frac{m}{s}$$

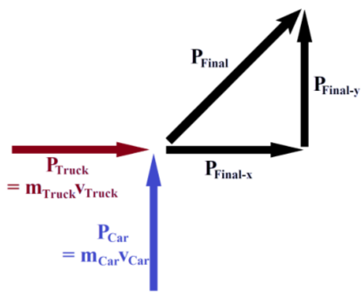
Is this an elastic or inelastic collision?

$$E_{init} = \frac{1}{2}mv_0^2 = \frac{1}{2}m\left(3.00\frac{m}{s}\right)^2 = \left(4.50\,\frac{m^2}{s^2}\right)m$$

$$E_{final} = \frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2 = mv_A^2 = m\left(2.12132\frac{m}{s}\right)^2 = \left(4.50\,\frac{m^2}{s^2}\right)m$$

So it's elastic.

<u>Example</u>: A car (m_{car} = 1300 kg) is heading north at 20.0 m/s when it collides with a truck (m_{truck} = 2000 kg) heading east at 15.0 m/s. During the collision the bumpers lock, holding the car and truck together. Determine the velocity of the pair after the collision.



$$P_{Final-x} = m_{Truck}v_{Truck} = (2000\,kg)\left(15.0\frac{m}{s}\right) = 30{,}000\,kg\cdot\frac{m}{s}$$

$$P_{Final-y} = m_{car}v_{car} = (1300\,kg)\left(20.0\frac{m}{s}\right) = 26{,}000\,kg\cdot\frac{m}{s}$$

$$P_{Final} = \sqrt{P_{Final-x}^2 + P_{Final-y}^2} = \sqrt{\left(30{,}000\,kg\cdot\frac{m}{s}\right)^2 + \left(26{,}000\,kg\cdot\frac{m}{s}\right)^2} = 39{,}699\,kg\cdot\frac{m}{s}$$

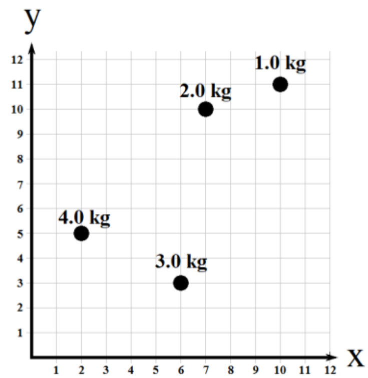
$$v_{Final} = \frac{P_{Final}}{m} = \frac{39{,}699\,kg\cdot\frac{m}{s}}{(2000\,kg + 1300\,kg)} = 12.0\frac{m}{s}$$

<u>Center of Mass</u>

- The dynamics of any object are equivalent to having the entire mass at a single point, the center of mass.
  - This allows us to treat every object as a point with mass.
  - As this is also true for gravity, the center of mass is also called the center of gravity.
- The Center of Mass of an object is the mean (average) position of its mass.
  - For scattered point masses:

$$x_{CM} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \qquad y_{CM} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}$$

Example: Find the center of mass.



$$x_{CM} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4}{m_1 + m_2 + m_3 + m_4}$$

$$x_{CM} = \frac{(4.0\,kg)(2.0\,m) + (3.0\,kg)(6.0\,m) + (2.0\,kg)(7.0\,m) + (1.0\,kg)(10.0\,m)}{4.0\,kg + 3.0\,kg + 2.0\,kg + 1.0\,kg} = 5.0\,m$$

$$y_{CM} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + m_4y_4}{m_1 + m_2 + m_3 + m_4}$$

$$y_{CM} = \frac{(4.0\,kg)(5.0\,m) + (3.0\,kg)(3.0\,m) + (2.0\,kg)(10.0\,m) + (1.0\,kg)(11.0\,m)}{4.0\,kg + 3.0\,kg + 2.0\,kg + 1.0\,kg} = 6.0\,m$$

- For mass distributions:

$$x_{CM} = \frac{\int x dm}{\int dm} \qquad y_{CM} = \frac{\int x dm}{\int dm}$$

- The object is broken into infinitesimally small pieces where 'dm' is the mass of any given piece. These are then added together (integration).
- Typically, dm is written in terms of volume and density.

$$m = \rho \cdot V \quad \text{therefore…} \quad dm = \rho \cdot dV$$

Example: Find the x-component of the center of mass of the triangle shown. Assume uniform thickness and density.

Step 1: Split mass into small pieces, each with the same value of x (as x appears in our equation)

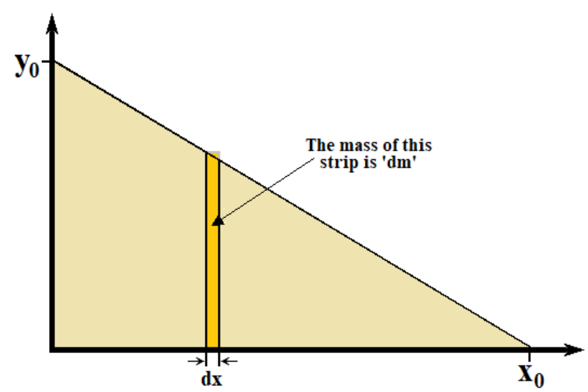
Step 2: Find 'dm', the mass of the strip.

$$dm = \rho dV = \rho z_0 A = \rho z_0 y dx$$

In this instance, we must find y as a function of x (since y varies with x).

$$y = mx + b \qquad m = \frac{\Delta y}{\Delta x} = \frac{0 - y_0}{x_0 - 0} = -\frac{y_0}{x_0}$$

$$b = y_0 \qquad dm = \rho z_0 y dx = \rho z_0 (mx + b)dx$$



Step 3: Plug in and integrate.

$$x_{CM} = \frac{\int x dm}{\int dm} = \frac{\int_0^{x_0} x[\rho z_0(mx+b)dx]}{\int_0^{x_0} \rho z_0(mx+b)dx} = \frac{\rho z_0 \int_0^{x_0}(mx^2+bx)dx}{\rho z_0 \int_0^{x_0}(mx+b)dx} = \frac{\int_0^{x_0}(mx^2+bx)dx}{\int_0^{x_0}(mx+b)dx}$$

$$x_{CM} = \frac{\left[\frac{1}{3}mx^3 + \frac{1}{2}bx^2\right]_0^{x_0}}{\left[\frac{1}{2}mx^2 + bx\right]_0^{x_0}} = \frac{\frac{1}{3}mx_0^3 + \frac{1}{2}bx_0^2}{\frac{1}{2}mx_0^2 + bx_0} = \frac{2mx_0^3 + 3bx_0^2}{3mx_0^2 + 6bx_0} = \frac{2\left(-\frac{y_0}{x_0}\right)x_0^3 + 3y_0x_0^2}{3\left(-\frac{y_0}{x_0}\right)x_0^2 + 6y_0x_0}$$

$$x_{CM} = \frac{-2y_0x_0^2 + 3y_0x_0^2}{-3y_0x_0 + 6y_0x_0} = \frac{y_0x_0^2}{3y_0x_0} = \frac{1}{3}x_0$$