

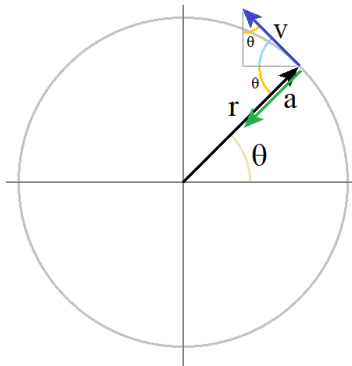
## Lecture 17: Simple Harmonic Motion, Wave Motion, and Sound

*Physics for Engineers & Scientists (Giancoli): Chapters 14, 15, & 16*

*University Physics VI (Openstax): Chapters 15, 16, & 17*

### Simple Harmonic Motion vs. Uniform Circular Motion

- The x- and y-components of uniform circular motion are in simple harmonic motion.
  - Simple harmonic motion can be viewed as a 1-dimensional view of uniform circular motion.
  - Uniform circular motion can be viewed as 2-dimensional simple harmonic motion where the two components are  $90^\circ$  out of phase.



*Let the radius of the circle be the amplitude (A).  $r = A$*

*There is no angular acceleration.  $\theta = \omega t + \theta_0$*

$$x = r \cos \theta = A \cos(\omega t + \theta_0)$$

$$v_x = -v \sin \theta = -\omega r \sin(\omega t + \theta_0) = -\omega A \sin(\omega t + \theta_0)$$

$$a_x = -a \cos \theta = -\frac{v^2}{r} \cos(\omega t + \theta_0) = -\frac{(\omega r)^2}{r} \cos(\omega t + \theta_0)$$

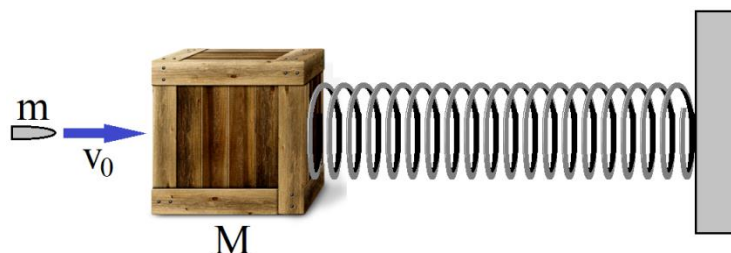
$$a_x = -\omega^2 r \cos(\omega t + \theta_0) = -\omega^2 A \cos(\omega t + \theta_0)$$

**Example:** Military aircraft and pilots are tested to ensure they can withstand accelerations of  $9g$  ( $88.2 \text{ m/s}^2$ ). To ensure that communication equipment can withstand these g-forces it is placed on an oscillating table that shifts back and forth in simple harmonic motion at a frequency of  $5.25 \text{ Hz}$ . To ensure that the equipment is tested at a maximum acceleration of  $9g$ , what amplitude is needed?

$$a_{\max} = \omega^2 A = (2\pi f)^2 A = 4\pi^2 f^2 A = 9g$$

$$A = \frac{9g}{4\pi^2 f^2} = \frac{9 \left( 9.80 \frac{\text{m}}{\text{s}^2} \right)}{4\pi^2 (5.25 \text{ Hz})^2} = 25.5 \text{ cm}$$

**Example:** A  $50.0 \text{ kg}$  block is attached to a spring ( $k = 450 \text{ N/m}$ ), which in turn is attached to a wall. The block is at rest when it is struck by a bullet with a trajectory that would pass straight down the center of the spring. The bullet becomes lodged in the block, and sends it into simple harmonic motion with frequency of  $0.4765 \text{ Hz}$  and amplitude  $50.5 \text{ cm}$ . Determine the mass and the initial velocity of the bullet.



$$\omega = \sqrt{\frac{k}{m+M}} = 2\pi f \quad \frac{k}{m+M} = 4\pi^2 f^2 \quad m + M = \frac{k}{4\pi^2 f^2}$$

$$m = \frac{k}{4\pi^2 f^2} - M = \frac{\left(450 \frac{N}{m}\right)}{4\pi^2 (0.4765)^2} - 50.0 \text{ kg} = 0.20269 \text{ kg}$$

*Conservation of momentum relates the bullet velocity ( $v_0$ ) to  $V$ .*

$$mv_0 = (m + M)V \quad v_0 = \left(1 + \frac{M}{m}\right)V$$

*Conservation of energy (after the collision) relates  $V$  to  $A$ .*

*Or...*

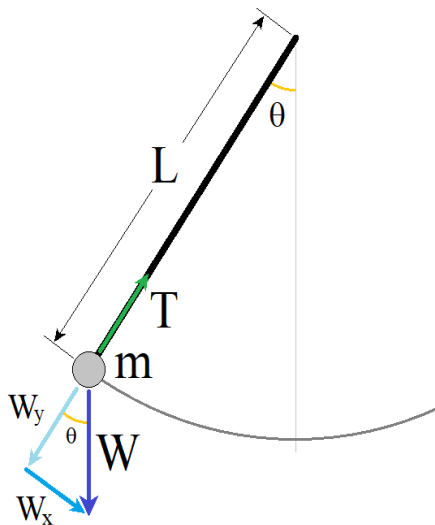
*The velocity of the block and bullet ( $V$ ) right after the collision occurs at equilibrium.  
That means  $V$  is the maximum velocity.*

$$V = v_{\max} = \omega A = 2\pi f A$$

$$v_0 = \left(1 + \frac{M}{m}\right)V = \left(1 + \frac{M}{m}\right)2\pi f A = \left(1 + \frac{50.0 \text{ kg}}{0.20269 \text{ kg}}\right)2\pi(0.4765 \text{ Hz})(0.505 \text{ m}) = 374 \frac{m}{s}$$

## Pendulum

- We will treat this as 1-dimensional simple harmonic motion along the arc made by the hanging mass.



Weight ( $W$ ) and Tension ( $T$ ) act on the hanging mass.

One component of the weight ( $W_y$ ) cancels out the tension.

The other component of the weight ( $W_x$ ) acts as the restoring force.

If we can find  $k$ , then we can use  $\omega = \sqrt{\frac{k}{m}}$

$$k = -\frac{F}{x} = -\frac{W_x}{-L\theta} = \frac{mg \sin \theta}{L\theta} \approx \frac{mg\theta}{L\theta} = \frac{mg}{L}$$

*In the small angle approximation,  $\sin \theta \approx \theta$*

$$\omega = \sqrt{\frac{mg}{Lm}} = \sqrt{\frac{g}{L}} \quad f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{g}{L}} \quad T = \frac{1}{f} = 2\pi \sqrt{\frac{L}{g}}$$

- The period/frequency of a pendulum is independent of mass.

**Example:** The pendulum in Big Ben has a 299 kg bob and a period of 2 seconds. What is the length of the arm of this pendulum?

$$T = 2\pi\sqrt{\frac{L}{g}} \quad \frac{T}{2\pi} = \sqrt{\frac{L}{g}} \quad \frac{T^2}{4\pi^2} = \frac{L}{g} \quad L = \frac{gT^2}{4\pi^2} = \frac{(9.80\frac{m}{s^2})(2.00\text{ s})^2}{4\pi^2} = 99.3\text{ cm}$$

**Example:** Wilson Hall, the picturesque administrative building at Fermilab, used to have a rather slow-moving pendulum hanging from the very top of the building, 16 floors high (roughly 160 ft.). How long does it take this pendulum to make one complete cycle?

$$L = (160\text{ ft.}) \frac{(0.3048\text{ m})}{(1\text{ ft.})} = 48.768\text{ m}$$

$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{48.768\text{ m}}{9.80\frac{m}{s^2}}} = 14.0\text{ s}$$

## Damping, Driving, and Resonance

- Springs are useful for absorbing impacts.
- Choosing the spring constant allows smaller deceleration to occur over a larger distance, reducing the force of impact.
- Springs absorb and release a portion of the impact energy, which can lead to unwanted harmonic motion.
- In **Damped Harmonic Motion**, energy is steadily removed from the system resulting in decreasing amplitude.

*Cars are suspended on springs so that when you drive over a bump the deceleration is gentler. If the absorbed collision energy is not dissipated by damping (shock absorbers), then your vehicle would continue to bounce.*

- In **Driven Harmonic Motion**, energy is added from an outside source.
  - How a driven harmonic oscillator behaves is dependent upon both the frequency of the driving force and the natural frequency of the oscillator.
  - **Resonance** occurs when the driving frequency and the oscillator frequency match. When this happens, energy is continually added to the system.

*When pushing a child in a swing, timing your pushes to the timing of the swing results in the child swinging higher and higher (ever increasing amplitude) even as the friction in the system (damping) causes their swinging to slow down and lose height.*

## The Wave Equation and its Solutions

- In some instances, Newton's laws lead to a partial differential equation known as the Wave Equation.

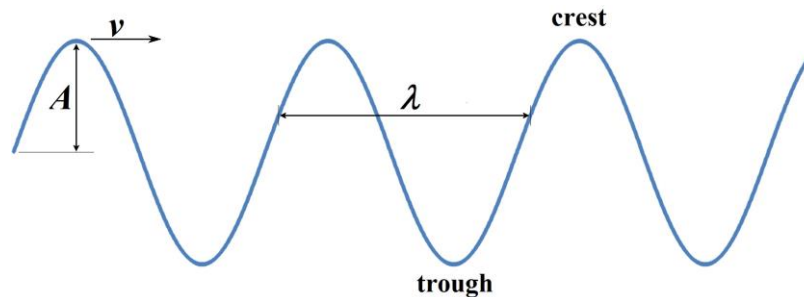
$$\frac{\partial^2 D}{\partial t^2} - v^2 \frac{\partial^2 D}{\partial x^2} = 0$$

*D = Displacement as measured from equilibrium.*

*The constant ( $v^2$ ) is the square of the velocity of the wave.*

- The solutions to this equation are travelling waves, either a sine function, a cosine function, or a combination of the two depending upon the initial phase.

$$D = A \sin(kx - \omega t) = A \sin\left[2\pi\left(\frac{x}{\lambda} - ft\right)\right] = A \sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right] = A \sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$



- The Wave Number ( $k$ ) is the spatial frequency of the wave (cycles per unit distance).

$$k = \frac{2\pi}{\lambda} \quad v = \frac{\omega}{k} = \frac{2\pi f}{\frac{2\pi}{\lambda}} = f\lambda = \frac{\lambda}{T}$$

- The velocity ( $v$ ) is determined by the properties of the medium through which the wave moves.
- The frequency ( $f$ ), angular frequency ( $\omega$ ), period ( $T$ ), wavelength ( $\lambda$ ), and wave number ( $k$ ) are all inter-related and determined by whatever excitation created the wave.
- Travelling waves obey Superposition, meaning that the displacements of two different waves simply add together, superimposing one wave on top of the other.
- When waves of the same frequency combine (superimpose), it is called Interference and can create a resultant wave of greater (Constructive Interference) or lower (Destructive Interference) amplitude.
- Driven travelling waves may also experience Resonance.
- Travelling waves are Longitudinal if the movement of particles making the wave is parallel/anti-parallel with the direction of the wave's motion.
- Travelling waves are Transverse if the movement of particles making the wave is perpendicular to the direction of the wave's motion.

- Travelling waves often reflect back when they encounter boundaries and may invert (180° phase shift) upon reflection.

**Example:** The amplitude of an ocean swell is 1.50 m with crests separated 33.8 m. A wave crest strikes the beach once every 5.70 s. Determine (A) the frequency of the waves, (B) the speed of the waves, and (C) the wave number.

$$A = 1.50 \text{ m} \quad \lambda = 33.8 \text{ m} \quad T = 5.70 \text{ s.}$$

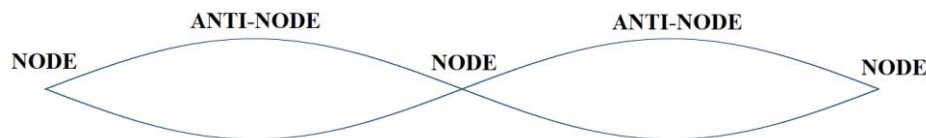
$$f = \frac{1}{T} = \frac{1}{5.70 \text{ s}} = 0.175 \text{ Hz} \quad v = \frac{\lambda}{T} = \frac{33.8 \text{ m}}{5.70 \text{ s}} = 5.93 \frac{\text{m}}{\text{s}} \quad k = \frac{2\pi}{\lambda} = \frac{2\pi}{33.8 \text{ m}} = 0.186 \text{ m}^{-1}$$

## Transverse Waves on Strings

- The velocity of waves on a string are given by:  $v = \sqrt{\frac{F_T}{\mu}}$ 
  - $F_T$  is the tension in the string (so as not to be confused with the period T).
  - $\mu$  is the mass per unit length of the string.

*The length of a string determines the wavelengths allowed. If we assume  $\lambda$  is fixed, then the frequency of the wave increases with the velocity.  $f = v/\lambda$ .*

  - Increasing the tension (typically by turning a tuning nut) increases the velocity of the waves, creating higher pitched notes.
  - Thicker strings have higher values of  $\mu$ , and higher values of  $\mu$  lead to lower velocities and lower pitched notes.
- Waves travelling down a string will reflect back from the ends of the string interfering with the original waves.
  - The majority of frequencies experience destructive interference.
  - Only a few specific modes of vibration (also known as **Harmonics**) experience constructive interference. In these cases, the superposition of the original wave and its reflections results in a **Standing Wave**, a wave that oscillates in time at amplitudes that are fixed in space.



- In each mode, certain positions called on the string **Nodes** have zero amplitude (no vibration). As the ends of strings are typically held in place (as the string is under tension), these must become nodes.
- Certain other positions called **Anti-Nodes** are vibrating with the maximum amplitude.

*All modes begin vibrating when a string is plucked. Normally the amplitudes of these various modes fall as you move to higher harmonics (making the higher modes significantly quieter). The various amplitudes are also affected by how and where a string is struck or plucked. The sum total of all of these modes produces the sound that you hear.*

- Each mode of vibration ( $n$ ) occurs at a specific wavelength (and frequency) related to the string's length ( $L$ ).

$$\lambda_n = \frac{2L}{n} \quad f_n = \frac{v_0}{\lambda_n} = \frac{nv_0}{2L} \quad \#Nodes = n + 1 \quad \#Anti-Nodes = n$$

- The first harmonic or fundamental mode ( $n = 1$ ) is the simplest and usually the loudest tone heard (largest amplitude). It has a node at each end, and one anti-node in between.

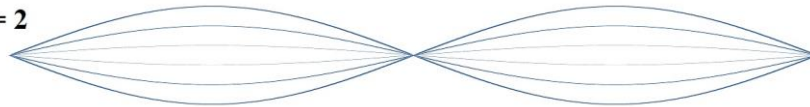
**n = 1**



$$L = \frac{1}{2}\lambda_1 \quad \lambda_1 = 2L \quad f_1 = \frac{v_0}{\lambda_1} = \frac{v_0}{2L}$$

- The second harmonic ( $n = 2$ ) has a node at each end, another node in the middle, and two anti-nodes (halfway between each adjacent pair of nodes).

**n = 2**



$$\lambda_2 = L \quad f_2 = \frac{v_0}{\lambda_2} = \frac{v_0}{L}$$

- The third harmonic ( $n = 3$ ) has a node at each end, two more nodes evenly spaced in between, and three anti-nodes (halfway between each adjacent pair of nodes).

**n = 3**



$$L = \frac{3}{2}\lambda_3 \quad \lambda_3 = \frac{2L}{3} \quad f_3 = \frac{v_0}{\lambda_3} = \frac{3v_0}{2L}$$

- The fourth harmonic ( $n = 4$ ) has 5 nodes and 4 anti-nodes.

**n = 4**



$$L = 2\lambda_4 \quad \lambda_4 = \frac{L}{2} \quad f_4 = \frac{v_0}{\lambda_4} = \frac{2v_0}{L}$$

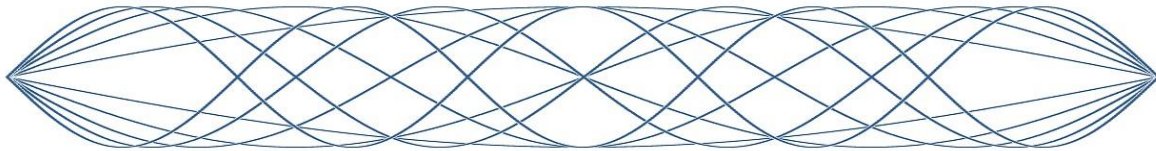
*Sometimes a guitarist will play a "harmonic" by holding their finger gently on the string without pressing it against the fret board (often to tune). This forces a node at that position, suppressing all modes of vibration that don't have this node (including the fundamental). If done at the 5<sup>th</sup> fret sounds are dominated by the 4<sup>th</sup> harmonic mode. The 7<sup>th</sup> fret gives you the 3<sup>rd</sup> harmonic mode, and the 12<sup>th</sup> fret gives you the second.*

- The fifth harmonic ( $n = 5$ ) has 6 nodes and 5 anti-nodes.



$$L = \frac{5}{2} \lambda_5 \quad \lambda_5 = \frac{2L}{5} \quad f_5 = \frac{v_0}{\lambda_5} = \frac{5v_0}{2L}$$

*When the envelopes of the first five harmonics of a string are superimposed, it makes a rather striking image.*

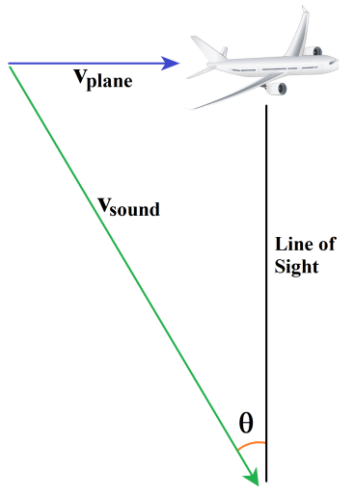


## Sound Waves

- Sound is a pressure wave (also called a compression wave).
- The velocity of sound waves in liquid or gas:  $v_{\text{sound}} = \sqrt{\frac{\beta}{\rho}}$ 
  - $\beta$  is the bulk modulus and  $\rho$  is the mass density of the medium.
  - The speed of sound in air varies with temperature. At 20°C:  $v_{\text{sound}} = 343 \frac{\text{m}}{\text{s}}$
  - The speed of sound in water depends on pressure (depth), temperature, and salinity. On average the speed of sound is 1560 m/s in saltwater and 1435 m/s in freshwater.
  - In liquids and gases, sound is strictly a longitudinal wave with alternating bands of high pressure (crests) and low pressure (troughs).
- The velocity of sound waves in a solid:  $v_{\text{sound}} = \sqrt{\frac{Y}{\rho}}$ 
  - $Y$  is the Young's modulus and  $\rho$  is the mass density of the medium.
  - The speed of sound in iron is roughly 5130 m/s.
  - In solids sound can propagate as either a longitudinal or a transverse wave. The transverse waves are alternating shear stress at a right angle to the propagation, and the longitudinal waves are alternating bands of high pressure (crests) and low pressure (troughs).

*Seismic activity (earthquakes) generates both (primary) longitudinal waves (P-waves) and (secondary) transverse waves (S-waves). The longitudinal waves travel at roughly 5000 m/s (in granite) while the slower transverse waves travel at only 3000 m/s. Typically the transverse waves do greater damage as they are typically created with larger amplitudes.*

**Example:** As a plane flies overhead you notice that the sound of the engines appears to be coming from a spot  $20.0^\circ$  behind the aircraft. How fast is the airplane moving?



$$v_{airplane} = v_{sound} \sin \theta = \left(343 \frac{m}{s}\right) \sin 20.0^\circ = 117 \frac{m}{s}$$

*117 m/s is about 262 mph.*

*At altitudes below 10,000 ft., aircraft are limited to a maximum speed of 250 knots (288 mph).*