Lecture 16: Fluids and Simple Harmonic Motion

Physics for Engineers & Scientists (Giancoli): Chapters 13 & 14
University Physics V1 (Openstax): Chapters 14 & 15

Gauge Pressure

- Some types of gauges (such as those that measure the air pressure in tires) only measure the difference between the pressure measured and atmospheric pressure.
    - The measured value is called the **gauge pressure**.
    - The actual pressure is called the **absolute pressure**.

$$P_{Absolute} = P_{Gauge} + P_{atm}$$

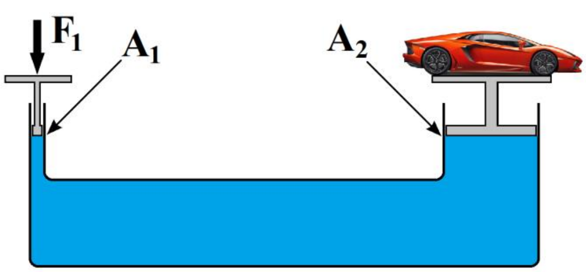
The barometer with the column of fluid supported by pressure measures the absolute pressure.

The pressure with depth equation is valid with both absolute and gauge pressures. However, most equations are only valid with absolute pressure.

Pascal's Principle

"Any change in pressure applied to a completely enclosed fluid is transmitted undiminished to every part of the fluid and to the walls enclosing it."

This is a direct result of $P_2 = P_1 + \rho g h$.
Any change to P_1 *changes* P_2 *as well, and* P_2 *could be anywhere in the fluid.*



The two pistons are at the same height. They must have the same pressure.

$$P_1 = P_{atm} + \frac{F_1}{A_1} \qquad P_2 = P_{atm} + \frac{F_2}{A_2}$$

$$\frac{F_1}{A_1} = \frac{F_2}{A_2} \qquad F_1 = \frac{A_1}{A_2} F_2$$

The mechanical advantage created in this way is the basis of hydraulics.

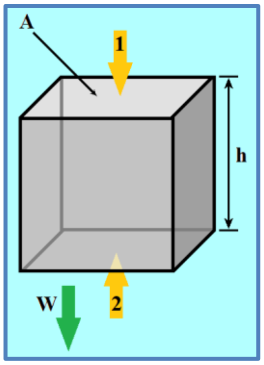
Example: The input piston of a hydraulic lift in an automotive repair shop has a radius of 1.00 cm. The output piston has a radius of 32.0 cm. What force is needed at the input piston to lift the 1570 kg car on the output piston?

$$F_1 = \frac{A_1}{A_2} F_2 = \frac{\pi r_1^2}{\pi r_2^2} F_2 = \left(\frac{r_1}{r_2}\right)^2 F_2 = \left(\frac{r_1}{r_2}\right)^2 mg = \left(\frac{1.00\ cm}{32.0\ cm}\right)^2 (1570\ kg)\left(9.80\frac{m}{s^2}\right) = 15.0\ N$$

Energy is still conserved. For every millimeter the car moves, the input piston must move a meter.

Archimedes' Principle "The buoyant force is equal to the weight of the fluid displaced."

- The net sum of the force of pressure on all sides of an object is called the Buoyant Force, and it points upward.



Let's take a large mass of a static fluid and place and object inside. We can calculate the buoyant force on this object.

$$F_B = F_2 - F_1 = P_2 A - P_1 A = [P_2 - P_1]A =$$

$$F_B = [(P_1 + \rho_F gh) - P_1]A = \rho_F ghA = \rho_F gV =$$

$$F_B = m_{DF} g = W_{DF}$$

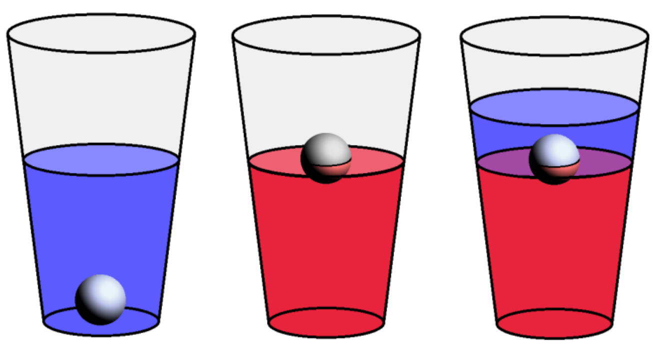
Archimedes' Principle: The buoyant force on an object is equal to the weight of the fluid displaced.

If the object is replaced with fluid (the same fluid), the fluid filling the boundaries is static. If the fluid is static, then the net force must be zero. If the net force is zero, then the buoyant force must equal to the weight. The buoyant force doesn't change when we return the original object.

- To get the Apparent Weight (W_{App}) of an object subtract the buoyant force from the weight.

Objects in water are easier to lift as the buoyant force helps us.

Conceptual Example: If blue fluid is added to the glass with red fluid, will the ball rise or fall?



The grey ball sinks to the bottom when it is placed in the glass with the blue fluid.

The grey ball floats with half of its volume submerged when placed in the glass with the red fluid.

If the blue fluid is slowly added to the top of the red fluid (with the ball floating on top) will the ball rise or fall?

Before the blue fluid is added, the ball has half of its volume submerged.

$$F_B = W_{DF} = \rho_{RF}\left(\tfrac{1}{2}V_{Ball}\right)g + \rho_{Air}\left(\tfrac{1}{2}V_{Ball}\right)g = \left(\tfrac{1}{2}\rho_{RF} + \tfrac{1}{2}\rho_{Air}\right)V_{Ball}\cdot g = W_{Ball}$$

If the ball remains in the same place (when the blue fluid is added), here is the buoyant force:

$$F_B = W_{DF} = \rho_{RF}\left(\tfrac{1}{2}V_{Ball}\right)g + \rho_{BF}\left(\tfrac{1}{2}V_{Ball}\right)g = \left(\tfrac{1}{2}\rho_{RF} + \tfrac{1}{2}\rho_{BF}\right)V_{Ball}\cdot g$$

As $\rho_{BF} > \rho_{Air}$, then: $\left(\tfrac{1}{2}\rho_{RF} + \tfrac{1}{2}\rho_{BF}\right)V_{Ball}\cdot g > \left(\tfrac{1}{2}\rho_{RF} + \tfrac{1}{2}\rho_{Air}\right)V_{Ball}\cdot g = W_{Ball}$

As the buoyant force is greater than the mass of the ball, the ball will rise until the buoyant force is equal to the weight.

Example: A duck is floating on the water with half of its volume underwater. Determine the density of the duck.

$$F_B = W_{Duck} = m_{Duck} \cdot g = \rho_{Duck} \cdot V_{Duck} \cdot g$$

$$F_B = W_{DF} = m_{DF} \cdot g = \rho_{DF} \cdot V_{DF} \cdot g = \rho_{DF} \cdot \left(\frac{1}{2} V_{Duck}\right) \cdot g = \frac{1}{2} \rho_{DF} \cdot V_{Duck} \cdot g$$

$$\rho_{Duck} \cdot V_{Duck} \cdot g = \frac{1}{2} \rho_{DF} \cdot V_{Duck} \cdot g$$

$$\rho_{Duck} = \frac{1}{2} \rho_{DF} = \frac{1}{2}\left(1000 \frac{kg}{m^3}\right) = 500 \frac{kg}{m^3}$$

Fluids in Motion

- In **Steady Flow**, all the particles are moving at the same speed as they pass a given point.
- In **Unsteady Flow**, particles are moving at different speeds as they pass a given point.
- In **Turbulent Flow**, velocities can change radically at a given point.
- A fluid is **Compressible** if the density of the fluid changes with pressure.
- A fluid is **Incompressible** if the density of the fluid is constant with changes with pressure.

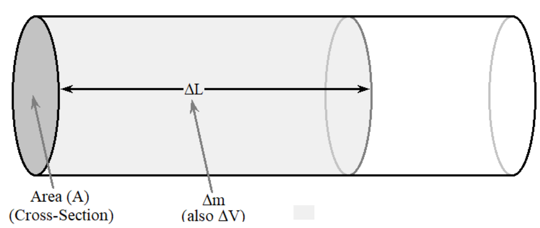
Fluids are nearly incompressible/ Gases are compressible.

- A **Viscous** fluid doesn't flow readily.
- A **Non-viscous** fluid flows readily.

We will assume that all fluids are Ideal Fluids, meaning they are incompressible and non-viscous. We will also assume that the flow is steady.

Mass Flow Rate

- **Mass Flow Rate** is the amount of mass that passes through a cross-section of the pipe in a given time interval.



$$MFR = \frac{\Delta m}{\Delta t} = \frac{\rho \Delta V}{\Delta t} = \frac{\rho A \Delta L}{\Delta t} = \rho A v$$

ΔV is the volume occupied by the mass that will move through the cross section. ΔL is the length of ΔV, and if this masses through the cross-section in Δt, then the velocity of the fluid must be ΔL/ Δt.

- Contained fluid (such as in a pipe) has a fixed volume. If we assume it is an ideal fluid with a constant density, then it also has a fixed mass. Any mass that enters one end implies that an equal mass leave the other end. In other words, the mass flow rate is constant.

$$\rho A_1 v_1 = \rho A_2 v_2$$

- Dividing out the density gives the **Volume Flow Rate**, which is also constant.

$$A_1 v_1 = A_2 v_2$$

Example: Water flows through a pipe of diameter 0.500 m at a velocity 0.250 m/s. The water flow is then constricted to a pipe of diameter 0.250 m. Determine A) the mass flow rate, and B) the velocity in the narrow pipe.

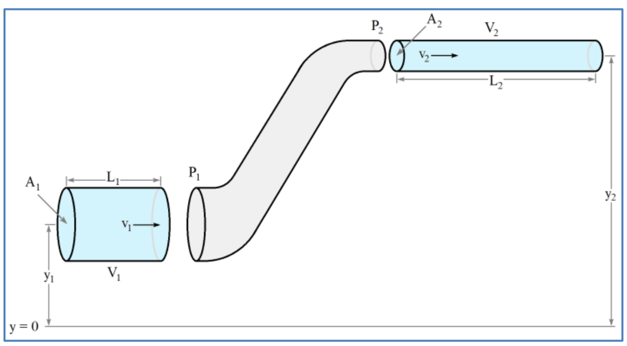
$$MFR = \rho A v = \left(1000\frac{kg}{m^3}\right)\pi\left(\frac{0.500\ m}{2}\right)^2\left(0.250\frac{m}{s}\right) = 49.1\frac{kg}{s}$$

$$A_1 v_1 = A_2 v_2$$

$$v_2 = \frac{A_1}{A_2}v_1 = \frac{\pi r_1^2}{\pi r_2^2}v_1 = \left(\frac{r_1}{r_2}\right)^2 v_1 = \left(\frac{d_1}{d_2}\right)^2 v_1 = \left(\frac{0.500\text{ m}}{0.250\text{ m}}\right)^2\left(0.250\frac{\text{m}}{\text{s}}\right) = 1.00\frac{m}{s}$$

The Bernoulli Equation

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$



If we push water into the pipe, an equal amount of water must come out the other side. (V_1=V_2)

If there is steady flow through the pipe, then energy must be conserved in this process.

The initial and final states have kinetic and gravitational potential energy, but energy is also added by work done by pressure.

$$E_{Init} = \frac{1}{2}mv_1^2 + mgy_1 = \frac{1}{2}\rho V_1 v_1^2 + \rho V_1 g y_1$$

$$E_{Final} = \frac{1}{2}mv_2^2 + mgy_2 = \frac{1}{2}\rho V_2 v_2^2 + \rho V_2 g y_2$$

$$E_{added} = W_1 + W_2 = F_1 L_1 - F_2 L_2 = P_1 A_1 L_1 - P_2 A_2 L_2 = P_1 V_1 - P_2 V_2$$

$$E_{Init} + E_{added} = E_{Final}$$

$$\frac{1}{2}\rho V_1 v_1^2 + \rho V_1 g y_1 + P_1 V_1 - P_2 V_2 = \frac{1}{2}\rho V_2 v_2^2 + \rho V_2 g y_2$$

$$\frac{1}{2}\rho v_1^2 + \rho g y_1 + P_1 - P_2 = \frac{1}{2}\rho v_2^2 + \rho g y_2$$

$$\boldsymbol{P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2}$$

Example: During a tornado the winds and pressure can be sufficient to rip the roofs off houses. The roof of a cabin is 10.0 m by 7.00 m. The air outside is moving at 67.0 m/s (roughly 150 mph). You may assume that the difference in height from the top of the roof to the bottom is negligible. The density of air is 1.225 kg/m^3. How much force is exerted on the roof?

$$F_{Net} = F_{inside} - F_{outside} = P_{inside}A - P_{outside}A = (P_{inside} - P_{outside})A = (P_1 - P_2)LW$$

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

As stated in the problem, $y_1 \approx y_2$ (ρgy terms cancel). Also, the air is still inside ($v_1 = 0$).

$$P_1 = P_2 + \frac{1}{2}\rho v_2^2$$

$$P_1 - P_2 = \frac{1}{2}\rho v_2^2 = \frac{1}{2}\left(1.225\frac{kg}{m^3}\right)\left(67.0\frac{m}{s}\right)^2 = 2{,}749.5\ Pa$$

$$F_{Net} = (P_1 - P_2)A = (2{,}749.5\ Pa)(10.0\ m)(7.00m) = 192\ kN$$

192 kN is roughly equal to 43,000 lbs.

Example: The top was removed from an old silo to convert it to collecting and storing rain water. The silo is 10.0 m high, 2.00 m in radius, and full to the open top with water. A poorly placed shot from a rifle puts a hole in the side of the silo that is 1.00 cm in diameter and 1.60 m above the ground. Determine the velocity of the water as it streams out the hole.

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

P_1 and P_2 are both 1 atm (open to the air) and will cancel.

$$\tfrac{1}{2}\rho v_1^2 + \rho g y_1 = \tfrac{1}{2}\rho v_2^2 + \rho g y_2 \qquad \rho g y_1 - \rho g y_2 = \tfrac{1}{2}\rho v_2^2 - \tfrac{1}{2}\rho v_1^2 \qquad \rho g(y_1 - y_2) = \tfrac{1}{2}\rho(v_2^2 - v_1^2)$$

Need to get v_1 (top of the silo) in terms of v_2 (coming out the bullet hole).

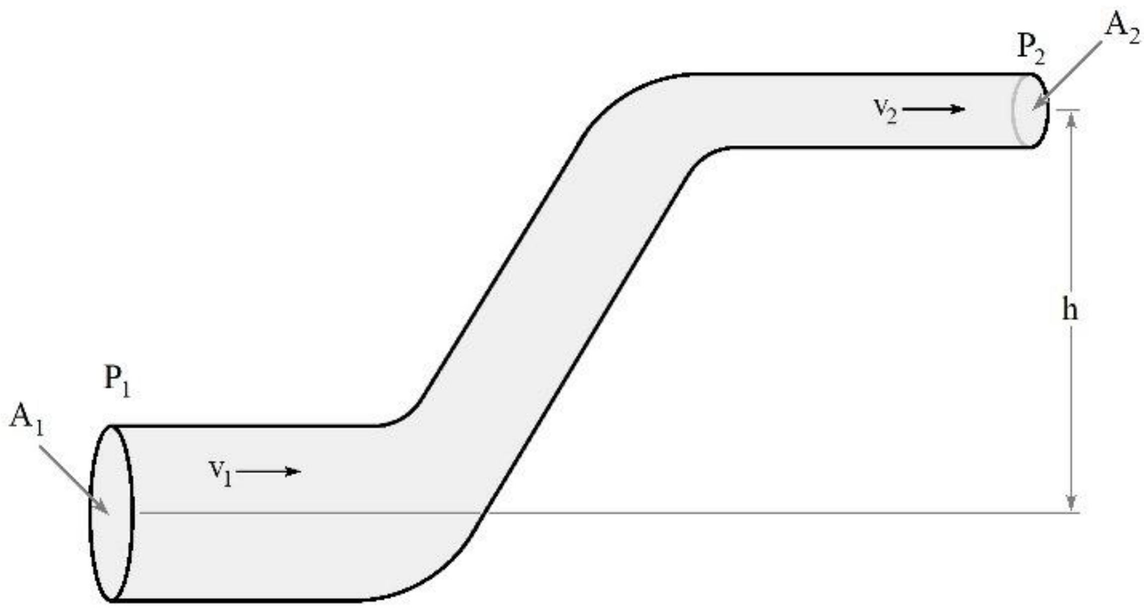
$$A_1 v_1 = A_2 v_2 \qquad v_1 = \frac{A_2}{A_1}v_2 = \frac{\pi r_2^2}{\pi r_1^2}v_2 = \left(\frac{r_2}{r_1}\right)^2 v_2 = \left(\frac{0.005\ m}{2.00\ m}\right)^2 v_2 = (6.25 \times 10^{-6})\ v_2$$

As v_1 is roughly a million times smaller than v_2, the v_1^2 is negligible compared to v_2^2. Keep this in mind. In many cases, one of the KE terms is negligible.

$$\rho g(y_1 - y_2) = \tfrac{1}{2}\rho v_2^2 \qquad v_2^2 = 2g(y_1 - y_2)$$

$$v_2 = \sqrt{2g(y_1 - y_2)} = \sqrt{2\left(9.80\frac{m}{s^2}\right)(10.0\ m - 1.60\ m)} = 12.8\frac{m}{s}$$

Example: In order to clear an obstacle, an oil pipeline rises h = 12.5 m in height. As it does the pipeline narrows from a diameter of 1.20 m to a diameter of 0.625 m. The density of the crude oil is 827 kg/m^3, and the velocity of the oil at the bottom is v_1 = 1.25 m/s. If the pressure at the top (P_2), can't exceed 56.7 atm, determine the maximum pressure of P_1.



$$A_1 v_1 = A_2 v_2 \qquad v_2 = \frac{A_1}{A_2} v_1 = \frac{\pi r_1^2}{\pi r_2^2} v_1 = \left(\frac{r_1}{r_2}\right)^2 v_1 = \left(\frac{d_1}{d_2}\right)^2 v_1 = \left(\frac{1.20\ m}{0.625\ m}\right)^2 \left(1.25\frac{m}{s}\right) = 4.608\frac{m}{s}$$

$$P_2 = 56.7\ atm\left(\frac{1.013\times 10^5\ Pa}{1\ atm}\right) = 5.74371\times 10^6\ Pa$$

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

$$P_1 = P_2 + \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(y_2 - y_1) = P_2 + \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g h$$

$$P_1 = 5.74371\times 10^6\ Pa + \frac{1}{2}\left(827\frac{\text{kg}}{\text{m}^3}\right)\left[\left(4.608\frac{m}{s}\right)^2 - \left(1.25\frac{m}{s}\right)^2\right] + \left(827\frac{\text{kg}}{\text{m}^3}\right)\left(9.80\frac{\text{m}}{\text{s}^2}\right)(12.5\text{m})$$

$$P_1 = 5.74371\times 10^6\ Pa + 8134\ Pa + 101{,}308\ Pa = 5.85\times 10^6\ Pa$$

Simple Harmonic Motion

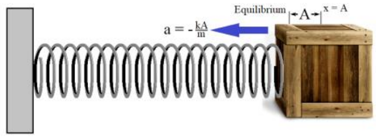
- Simple Harmonic Motion is a repetitive (periodic) state of motion that occurs when the magnitude of the restoring force is proportional to the displacement from equilibrium.

One place this type of behavior occurs is when a mass is attached to a spring and allowed to slide across a frictionless surface.

- If we pull the mass a distance A away from equilibrium, the Restoring Force of the spring will pull it back towards the equilibrium position.

$t = 0$

$t = nT$



$x = x_{max} = A$

$v = 0$

$a = -a_{max} = -\frac{kA}{m} = -\omega^2 A$

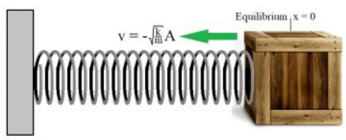
  - The Amplitude (A) is the maximum displacement of the system from equilibrium.
  - The mass is released from rest in this case. $v_0 = 0$.
  - The acceleration is: $a = \frac{F}{m} = \frac{-kx}{m} = \frac{-kA}{m}$
  - As will be shown later, the angular frequency (ω) of the system is: $\omega = \sqrt{\frac{k}{m}}$
  - The object will return to this exact state of the beginning of every period

$$t = nT, \quad for\ n = 0, 1, 2, \ldots$$

- After a quarter of the period, the mass will have returned to its equilibrium position.

$t = \frac{1}{4}T$

$t = (n + \frac{1}{4})T$



$x = 0$

$v = -v_{max} = -\sqrt{\frac{k}{m}}A = -\omega A$

$a = 0$

  - At equilibrium (x = 0) the net force is zero. This means there is no longer any acceleration.
  - One it passes equilibrium it will begin to decelerate, making this the maximum speed.
  - We can calculate that speed using conservation of energy:

$$E_{Init} = U_{Elastic} = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \qquad E_{Final} = KE = \frac{1}{2}mv^2$$

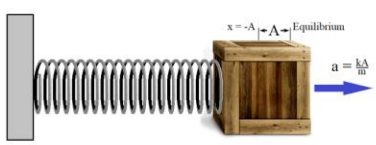
$$E_{Init} = E_{Final} \qquad \frac{1}{2}kA^2 = \frac{1}{2}mv^2 \qquad kA^2 = mv^2 \qquad \frac{k}{m}A^2 = v^2 \qquad v = \sqrt{\frac{k}{m}}A = \omega A$$

  - The object will return to this state every period: $t = (n + \frac{1}{4})T, \quad for\ n = 0, 1, 2, \ldots$

- After half of the period, the mass will have come to rest again.

$t = \frac{1}{2}T$

$t = (n + \frac{1}{2})T$



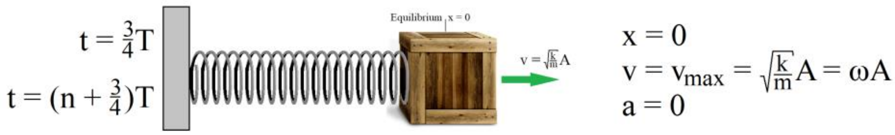
$x = -x_{max} = -A$

$v = 0$

$a = a_{max} = \frac{kA}{m} = \omega^2 A$

  - The amplitude has maximum magnitude again, this time on the negative side: x = -A.
  - The mass has come to rest again. $v_0 = 0$.
  - The acceleration is: $a = \frac{F}{m} = \frac{-kx}{m} = \frac{kA}{m}$
  - The object will return to this state every period: $t = (n + \frac{1}{2})T, \quad for\ n = 0, 1, 2, \ldots$

- After three quarters of the period, the mass will have returned to its equilibrium position again.

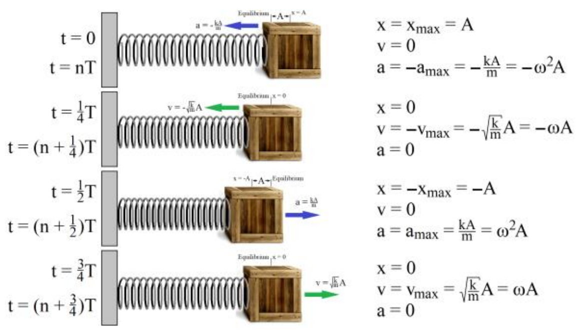


  - At equilibrium (x = 0) the net force is zero. This means there is no longer any acceleration.
  - One it passes equilibrium it will begin to decelerate, making this the maximum speed.

$$v = \sqrt{\frac{k}{m}}A = \omega A$$

  - The object will return to this state every period: $t = (n + \frac{3}{4})T, \quad for\ n = 0, 1, 2, \ldots$

- After one full period, the object returns to its initial position and state and the cycle begins again.



- The energy of the system continually changes from potential energy to kinetic energy and back.
- The velocity at any position can be calculated using conservation of energy:

$$E_{system} = E_{Init} = \frac{1}{2}kA^2 \qquad E_{Final} = U_{Elastic} + KE = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

$$E_{Init} = E_{Final} \qquad \frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \qquad \frac{1}{2}kA^2 - \frac{1}{2}kx^2 = \frac{1}{2}mv^2$$

$$\frac{1}{2}k(A^2 - x^2) = \frac{1}{2}mv^2 \qquad k(A^2 - x^2) = mv^2 \qquad \frac{k}{m}(A^2 - x^2) = v^2 \qquad v = \sqrt{\frac{k}{m}(A^2 - x^2)}$$

- <u>General Solution</u>

$$F = ma \qquad -kx = m\frac{d^2x}{dt^2} \qquad m\frac{d^2x}{dt^2} + kx = 0 \qquad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

The solution to this well-known differential equation is of the form: $x(t) = C_1 \cos \omega t + C_2 \sin \omega t$

The given initial position at t=0 is x=A. $x(0) = C_1 \cos 0° + C_2 \sin 0° = C_1 = A$

The given initial position is a maximum. $C_2 = 0$

$$x(t) = A\cos\omega t \qquad \frac{d^2x}{dt^2} = \frac{d^2}{dt^2}(A\cos\omega t) = \frac{d}{dt}(-\omega A\sin\omega t) = -\omega^2 A\cos\omega t$$

Plug into the initial different equation:

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad -\omega^2 A\cos\omega t + \frac{k}{m}A\cos\omega t = 0 \qquad \left(\frac{k}{m} - \omega^2\right)A\cos\omega t = 0$$

As $A \neq 0$ and $\cos(\omega t) \neq 0$ (at least it isn't for all values of t):

$$\frac{k}{m} - \omega^2 = 0 \qquad \omega = \sqrt{\frac{k}{m}} \qquad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Which gives us our solution: $x(t) = A\cos\omega t$ *with* $\omega = \sqrt{\frac{k}{m}}$

$$v = \frac{dx}{dt} = \frac{d}{dt}(A\cos\omega t) = -\omega A\sin\omega t \qquad v_{max} = \omega A = \sqrt{\frac{k}{m}}A$$

$$a = \frac{d^2x}{dt^2} = \frac{d^2}{dt^2}(A\cos\omega t) = -\omega^2 A\cos\omega t \qquad a_{max} = \omega^2 A = \frac{k}{m}A$$

This solution is only valid when the initial position is a maximum.