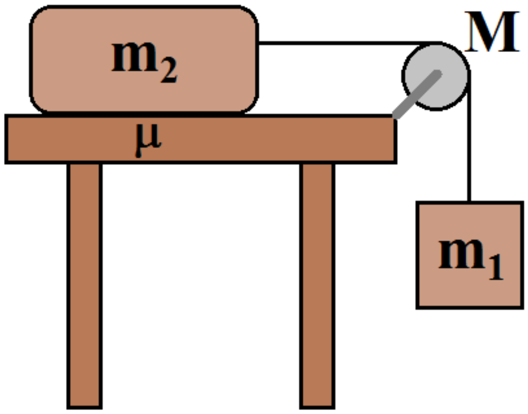
Lecture 14: Rotational Dynamics & Static Equilibrium

Physics for Engineers & Scientists (Giancoli): Chapters 11 & 12
University Physics V1 (Openstax): Chapters 10, 11, & 12



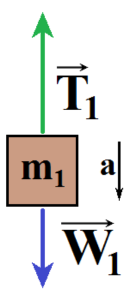
Example: A box of mass $m_2 = 10.0$ kg sits on a table as shown. On one side a cord connects it to a hanging weight of mass $m_1 = 25.0$ kg. The cord stretches over a frictionless pulley, a solid disk of uniform density with mass M = 5.00 kg. If the coefficient of kinetic friction between the box and table is 0.300, determine the acceleration of the box on the table.

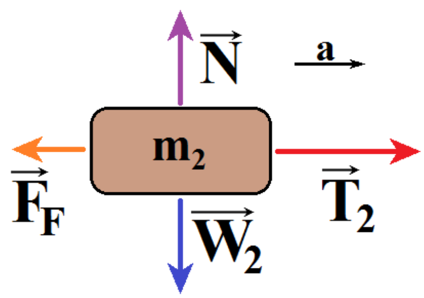
Make 3 force diagrams (one for each object). There are 3 unknowns (2 tensions and acceleration). This means you will need 3 equations, one from each force diagram. Angular acceleration is not another unknown as it is directly related to the acceleration in this problem.

The box on the table (m_2) accelerates to the right, which corresponds to a clockwise (CW) rotation of the pulley (M) and a downward acceleration of the hanging weight (m_1). To match the signs, for this problem we shall let CW rotations and downward accelerations become positive.

$$m_1 g - T_1 = m_1 a$$

$$T_1 = m_1 g - m_1 a$$



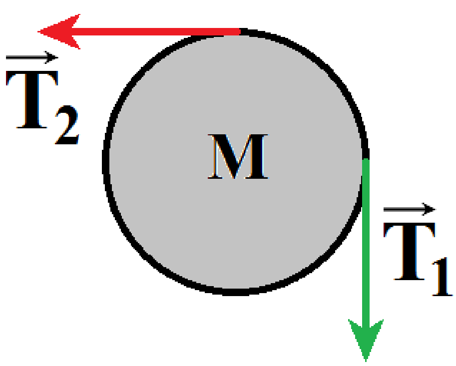


$$N = W_2 = m_2 g$$

$$F_F = \mu_k N = \mu_k m_2 g$$

$$T_2 - F_F = m_2 a$$

$$T_2 = m_2 a + \mu_k m_2 g$$



$$\sum \tau = \tau_1 + \tau_2 = RT_1 - RT_2 = R(T_1 - T_2)$$

$$\sum \tau = R(m_1 g - m_1 a - m_2 a - \mu_k m_2 g)$$

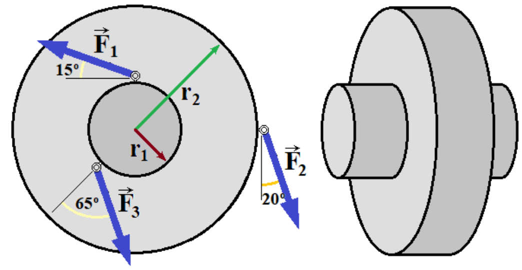
$$I\alpha = \left(\frac{1}{2}MR^2\right)\left(\frac{a}{R}\right) = \frac{1}{2}MRa \qquad I\alpha = \sum \tau$$

$$\frac{1}{2}MRa = R(m_1 g - m_1 a - m_2 a - \mu_k m_2 g)$$

$$\frac{1}{2}Ma = m_1 g - m_1 a - m_2 a - \mu_k m_2 g \qquad m_1 a + m_2 a + \frac{1}{2}Ma = m_1 g - \mu_k m_2 g$$

$$a = \frac{m_1 g - \mu_k m_2 g}{m_1 + m_2 + \frac{1}{2}M} = \frac{(m_1 - \mu_k m_2)g}{m_1 + m_2 + \frac{1}{2}M} = \frac{[(25.0\,kg) - (0.300)(10.0\,kg)](9.80\,m/s^2)}{25.0\,kg + 10.0\,kg + \frac{1}{2}(5.00\,kg)} = 5.75\frac{m}{s^2}$$

Example: Three forces act on a compound wheel as shown. The forces come from ropes tied to pins on the edges of the wheel. The moment of inertia of the wheel is 30.0 kg·m^2 with an inner radius of r_1= 25.0 cm and an outer radius of r_2 = 50.0 cm. Determine the angular acceleration of the wheel in response to the three forces: F_1 = 80.0 N, F_2 = 30.0 N, and F_3 = 20.0 N.

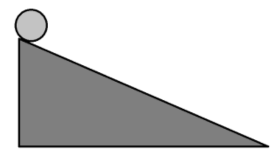


$$\sum \tau = \tau_1 + \tau_2 + \tau_3 = r_1 F_1 \sin 75° - r_2 F_2 \sin 70° + r_1 F_3 \sin 65°$$

$$\tau_{Net} = (0.250\ m)(80.0\ N) \sin 75° - (0.500\ m)(30.0\ N) \sin 70° + (0.250\ m)(20.0\ N) \sin 65°$$

$$\tau_{Net} = 9.7547\ N \cdot m \qquad \alpha = \frac{\tau_{Net}}{I} = \frac{9.7547\ N\cdot m}{30.0\ kg\cdot m^2} = 0.325 \frac{rad}{s^2}$$

Example: A hoop, a sphere, and a solid cylinder roll down an incline. Each has uniform density, the same mass and radius. If all three are released simultaneously, which gets to the bottom of the incline first?



We shall use $I = cMR^2$ as it applies to all 3 objects with the correct choice of c.
The object with the highest velocity at the bottom gets there first (highest V_{avg})

Conservation of Energy: $E_{init} = E_{Final}$ $\qquad mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

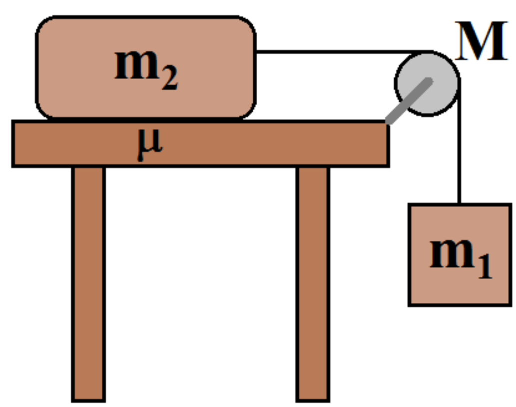
$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}(cMR^2)\left(\frac{v}{R}\right)^2 \qquad Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}cMv^2 = (1+c)\frac{1}{2}Mv^2$$

$$gh = (1+c)\frac{1}{2}v^2 \qquad v^2 = \frac{2gh}{1+c} \qquad v = \sqrt{\frac{2gh}{1+c}}$$

The object with the highest velocity at the bottom has the lowest value of 'c'.

The sphere wins because it has more mass near the axis of rotation.

Example: A box of mass m_2 = 10.0 kg sits on a table as shown. On one side a cord connects it to a hanging weight of mass m_1 = 25.0 kg. The cord stretches over a frictionless pulley, a solid disk of uniform density with mass M = 5.00 kg. The coefficient of kinetic friction between the box and table is 0.300. Determine the velocity of the hanging mass after it has fallen a distance of 0.500 m.



The previous similar problem asked for acceleration, which is related to forces. This problem asks for velocity, which is related to kinetic energy. Using conservation of energy is preferable.

The gravitational potential energy of the box (m_2) and the pulley (M) remain constant. We will ignore these as they will cancel out.

$$E_{init} - E_{Lost} = E_{Final} \qquad E_{init} = m_1 g h_0 \qquad E_{final} = \frac{1}{2}m_2 v^2 + \frac{1}{2}m_1 v^2 + \frac{1}{2}I\omega^2 + m_1 g h$$

$$E_{lost} = F_F d = \mu_k N d = \mu_k m_2 g d = \mu_k m_2 g(h_0 - h)$$

$$m_1 g h_0 - \mu_k m_2 g(h_0 - h) = \frac{1}{2}m_2 v^2 + \frac{1}{2}m_1 v^2 + \frac{1}{2}I\omega^2 + m_1 g h$$

$$m_1 g h_0 - m_1 g h - \mu_k m_2 g(h_0 - h) = \frac{1}{2}m_2 v^2 + \frac{1}{2}m_1 v^2 + \frac{1}{2}I\omega^2$$

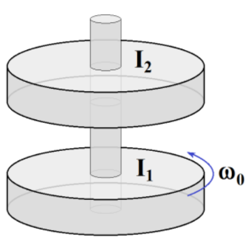
$$m_1 g(h_0 - h) - \mu_k m_2 g(h_0 - h) = \frac{1}{2}m_2 v^2 + \frac{1}{2}m_1 v^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2$$

$$(m_1 - \mu_k m_2)g(h_0 - h) = \frac{1}{2}m_2 v^2 + \frac{1}{2}m_1 v^2 + \frac{1}{4}Mv^2$$

$$4(m_1 - \mu_k m_2)g(h_0 - h) = 2m_2 v^2 + 2m_1 v^2 + Mv^2 = (2m_2 + 2m_1 + M)v^2$$

$$v^2 = \frac{4(m_1 - \mu_k m_2)g(h_0 - h)}{2m_2 + 2m_1 + M}$$

$$v = \sqrt{\frac{4(m_1 - \mu_k m_2)g(h_0 - h)}{2m_2 + 2m_1 + M}} = \sqrt{\frac{4[25.0\,kg - (0.300)(10.0\,kg)]\left(9.80\frac{m}{s^2}\right)(0.500m)}{2(10.0\,kg) + 2(25.0\,kg) + 5.00\,kg}} = 2.40\frac{m}{s}$$



<u>Example</u>: A solid disk ($I_1 = 4.00$ kg·m^2) is spinning about a fixed spindle at $\omega_0 = 15.0$ rad/s. A second solid disk ($I_2 = 6.00$ kg·m^2), which is not rotating, is placed on the spindle and dropped onto the first disk. There is friction between the two disks, and eventually they spin together. Determine (A) the velocity of the two discs once they start spinning together and (B) the energy is lost during the collision.

When spinning objects collide, it's a good indication that conservation of momentum will be relevant.

$$L_{init} = L_{Final} \qquad I_1\omega_0 = (I_1 + I_2)\omega \qquad \omega = \frac{I_1\omega_0}{I_1 + I_2} = \frac{(4.00\,kg\cdot m^2)\left(15.0\frac{rad}{s}\right)}{4.00\,kg\cdot m^2 + 6.00\,kg\cdot m^2} = 6.00\frac{rad}{s}$$

$$E_{Lost} = E_{Init} - E_{Final} = \frac{1}{2}I_1\omega_0^2 - \frac{1}{2}(I_1 + I_2)\omega^2$$

$$E_{Lost} = \frac{1}{2}(4.00\,kg\cdot m^2)\left(15.0\frac{rad}{s}\right)^2 - \frac{1}{2}(4.00\,kg\cdot m^2 + 6.00\,kg\cdot m^2)\left(6.00\frac{rad}{s}\right)^2 = 270\text{ J}$$

Example: An old park has a large turntable for children's play. It is initially at rest with a a radius of 1.20 m and a moment of inertia of 125 kg·m^2. A 50.0 kg woman runs at 8.00 m/s towards the edge of the turntable and jumps on, grabbing hold of the hand rail. Determine the angular velocity of the turntable after she jumps on.

This is also a conservation of angular momentum problem.

$$L_{Init} = L_{woman} = |\vec{r} \times \vec{p}| = |\vec{r} \times m\vec{v}| = Rmv \sin 90° = Rmv$$

$$L_{Final} = (I_{woman} + I_{turntable})\omega = (mR^2 + I)\omega \qquad (mR^2 + I)\omega = Rmv$$

$$\omega = \frac{Rmv}{mR^2 + I} = \frac{(1.20\,m)(50.0\,kg)\left(8.00\,\frac{m}{s}\right)}{(50.0\,kg)(1.20\,m)^2 + 125\,kg \cdot m^2} = 2.44\,\frac{rad}{s}$$

Example: The angular momentum of a precision grinding wheel as it starts to rotate is described by $L(t) = L_0(1-e^{-\beta t})$, with $L_0 = 315$ kg·m^2/s and $\beta = 0.247$ s^{-1}. Determine the net torque on the wheel at t = 3.17s.

$$\tau_{Net}(t) = \frac{dL}{dt} = \frac{d}{dt}\left[L_0\left(1 - e^{-\beta t}\right)\right] = \frac{d}{dt}\left[L_0 - L_0 e^{-\beta t}\right] = \beta L_0 e^{-\beta t}$$

$$\tau_{Net}(3.17\,s) = (0.247\,\mathrm{s}^{-1})\left(315\,\mathrm{kg} \cdot \frac{\mathrm{m}^2}{\mathrm{s}}\right) e^{-(0.247\,\mathrm{s}^{-1})(3.17\,s)} = 35.6\,N \cdot m$$

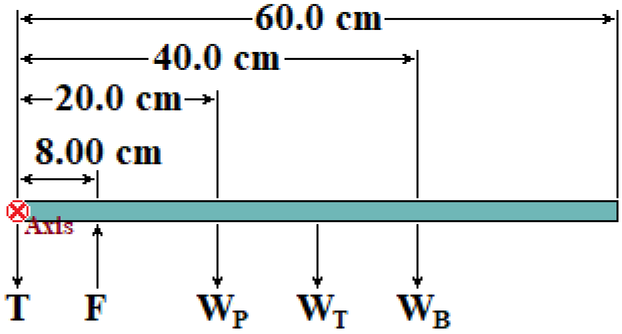
Statics

- To be in static equilibrium the net force on an object must be zero. In addition, the net torque must be zero as well.

$$\sum \vec{F} = 0 \quad \rightarrow \quad \begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \end{cases} \qquad \sum \vec{\tau} = 0$$

- Torque and moment of inertia are both defined about some axis of rotation. What do we use as an axis of rotation?
  - A static object is not accelerating about any axis. Therefore, the net torque about every axis is zero.
  - This gives us the freedom to select a convenient axis of rotation.
    - If we choose the point where a force acts to be the axis of rotation, then that force creates no torque (the lever arm is zero). This force will not show up in a torque equation.
    - A clever choice of axis of rotation can be used to remove a variable from your equation.

Example: A student carries a 60.0 cm long lunch tray with a single hand. To do so her fingers press upwards 8.00 cm from the left edge of the tray, and her thumb presses downward on the left edge. The mass of the lunch tray is 0.100 kg. They tray holds a bowl of soup of mass 0.500 kg whose center of mass sits 40.0 cm from the left edge and a plate of food of mass 0.750 kg whose center of mass sits 20.0 cm from the left edge. Determine the forces exerted by the student's thumb and fingers.



$$W_T = m_T g = (0.100\ kg)\left(9.80\frac{m}{s^2}\right) = 0.980\ N$$

$$W_B = m_B g = (0.500\ kg)\left(9.80\frac{m}{s^2}\right) = 4.90\ N$$

$$W_P = m_P g = (0.750\ kg)\left(9.80\frac{m}{s^2}\right) = 7.35\ N$$

Setting the sum of forces equal to zero will give us 1 equation with 2 unknowns. Consequently we will need to use a torque equation.

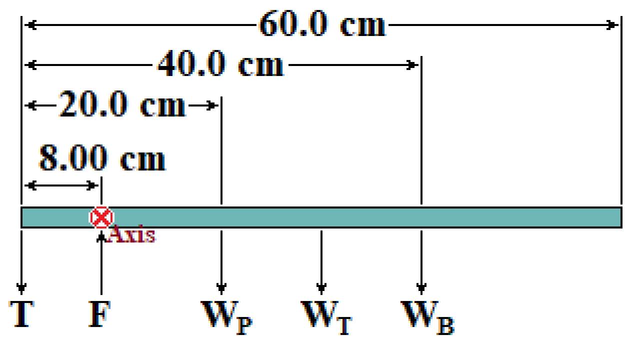
To start we will use the left edge as the axis of rotation. The force generated by the thumb (T) generates no torque about this axis and will disappear from our equations leaving just F.

$$\sum \tau = (8.00\ cm)F - (20.0\ cm)(7.35\ N) - (30.0\ cm)(0.980\ N) - (40.0\ cm)(4.90\ N) = 0$$

$$F = \frac{(20.0\ cm)(7.35\ N) + (30.0\ cm)(0.980\ N) + (40.0\ cm)(4.90\ N)}{8.00\ cm} = 46.55\ N$$

Once we have F we could use the sum of forces to get T, but for practice let's use another torque equation to get T.

To get T we need to move the axis of rotation. Let's place it at F.



$$\sum \tau = (8.00\ cm)T - (12.0\ cm)(7.35\ N) - (22.0\ cm)(0.980\ N) - (32.0\ cm)(4.90\ N) = 0$$

$$T = \frac{(12.0\ cm)(7.35\ N) + (22.0\ cm)(0.980\ N) + (32.0\ cm)(4.90\ N)}{8.00\ cm} = 33.3\ N$$