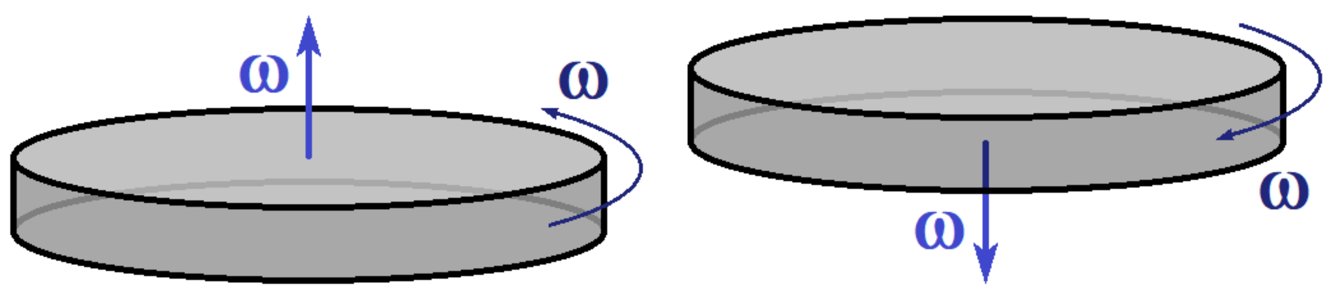
Lecture 13: Rotational Dynamics

Physics for Engineers & Scientists (Giancoli): Chapter 10
University Physics V1 (Openstax): Chapter 10

Rotational Vectors

- While we are able to treat rotational variables one-dimensionally in most cases, they are still vectors with magnitude and direction.
- The direction of rotational vectors is defined to be either parallel or anti-parallel to the axis of rotation. Anti-parallel means parallel but pointing the opposite direction.
- If the object is rotating counter-clockwise in an xy-plane when viewed from above, then ω points upward in the z-direction.
- If the object is rotating clockwise in an xy-plane when viewed from above, then ω points downward in the negative z-direction.



Rotational Dynamics

- As rotational kinematics showed many similarities with one-dimensional translational kinematics, we can expect more similarities to appear in dynamics, but there are some differences too.

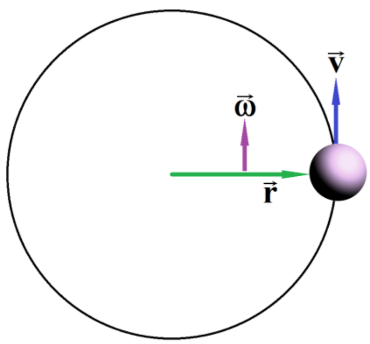
Concept	Translational (1D)	Rotational	Relationship
Position	x (or S)	θ	$S = r\theta$
Velocity	v	ω	$v = r\omega$
Acceleration	a	α	$a_T = r\alpha \quad a_R = \omega^2 r$
Cause of Acceleration	F	τ (Torque)	$\vec{\tau} = \vec{r} \times \vec{F} \quad (\lvert\tau\rvert = rF\sin\theta)$
Inertia	m	I (Moment of Inertia)	$dI = r^2 dm$
Newton's 2nd Law	$\vec{F} = m\vec{a}$	$\vec{\tau} = I\vec{\alpha}$	
Work	$W = Fd$	$W = \tau\theta$	
Kinetic Energy	$KE_{Translational} = ½mv^2$	$KE_{Rotational} = ½I\omega^2$	
Momentum	$\vec{P} = m\vec{v}$	$\vec{L} = I\vec{\omega}$	$\vec{L} = \vec{r} \times \vec{P}$
Force/Momentum	$\vec{F} = \frac{d\vec{P}}{dt}$	$\vec{\tau} = \frac{d\vec{L}}{dt}$	

<u>Moment of Inertia</u> (I)

- The moment of inertia is the rotational equivalent of mass (a resistance to being spun).

An object in uniform circular motion can be looked at as either rotational motion or as in linear translational motion (at least temporarily).

The kinetic energy of this motion should be the same either way it is calculated. This allows us to determine a relationship between moment of Inertia (I) and mass (m).



$$KE_{Translational} = \frac{1}{2}mv^2 = \frac{1}{2}m(r\omega)^2 = \frac{1}{2}mr^2\omega^2$$

$$KE_{Rotational} = \frac{1}{2}I\omega^2 \qquad \frac{1}{2}mr^2\omega^2 = \frac{1}{2}I\omega^2$$

$$I = mr^2$$

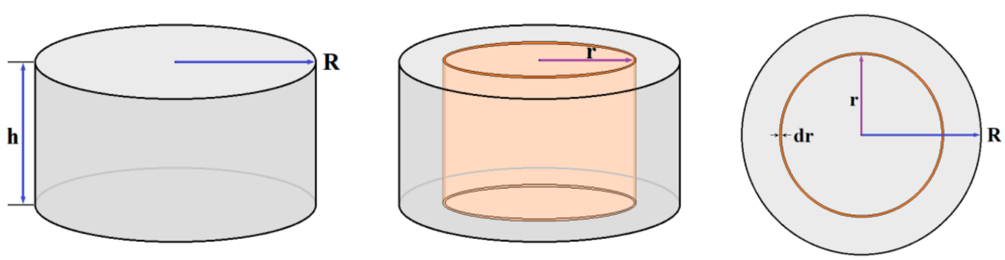
Note: This is only valid when all the mass is located at the same distance (r) from the axis of rotation.

- When the mass is not all at the same value of r, you must calculate the mass at each value of r and sum them all up. As there is often a continuous distribution of r values, summing may mean integration.

$$dI = r^2dm \qquad I = \int_{r_1}^{r_2} r^2dm \qquad dm = \rho dV$$

<u>Example</u>: Determine the moment of inertia of a solid cylindrical disk of uniform density, mass M, and radius R.

Step 1: Split mass into small pieces, each with the same value of r (as r appears in our equation). In this case, it will be infinitesimally thin cylindrical shells.



The left image shows the cylinder and its dimensions (R and h).
The center image shows one of the small pieces, a hollow cylinder in the center (orange).
The right image is a view from above. The thickness of the shell is dr.

Step 2: Find 'dm', the mass of a representative shell.

$$dm = \rho dV = \rho A dr = \rho(2\pi rh)dr = (2\pi\rho h)rdr \qquad \rho = \frac{M}{V} = \frac{M}{\pi R^2 h}$$

A is the surface area of the cylindrical shell. The length is the circumference and the width is h.

$$dm = (2\pi\rho h)rdr = \left(2\pi\frac{M}{\pi R^2 h}h\right)rdr = \frac{2M}{R^2}rdr$$

Be careful with your variables. Don't confuse R and r. R is the radius of the cylindrical disk. r is the radius of the thin cylindrical shell.

Step 3: Plug in and integrate.

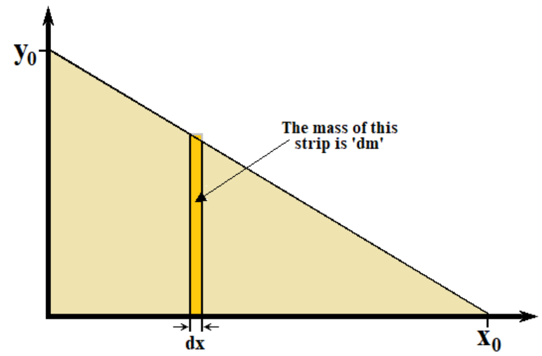
$$I = \int_0^R r^2 dm = \int_0^R r^2 \left(\frac{2M}{R^2} r dr\right) = \frac{2M}{R^2}\int_0^R r^3 dr = \frac{2M}{R^2}\left[\frac{r^4}{4}\right]_0^R = \frac{2M}{R^2}\left[\frac{R^4}{4}\right] = \frac{1}{2}MR^2$$

Example: Find the moment of inertia for a flat (right) triangular plate if it is rotating around the y-axis. You may assume uniform thickness and density.

Step 1: Split mass into small pieces, each with the same value of r (as r appears in our equation).

In this case the distance from the axis of rotation (the y-axis), is just x. (r = x)

Step 2: Find 'dm', the mass of the strip.



$$dm = \rho dV = \rho z_0 A = \rho z_0 y dx$$

In this instance, we must find y as a function of x (since y varies with x).

$$y = mx + b \qquad m = \frac{\Delta y}{\Delta x} = \frac{0 - y_0}{x_0 - 0} = -\frac{y_0}{x_0}$$

$$b = y_0 \qquad dm = \rho z_0 y dx = \rho z_0 (mx + b)dx$$

We also need the value of ρ. $\rho = \frac{M}{V} = \frac{M}{At} = \frac{M}{\frac{1}{2}bht} = \frac{2M}{bht} = \frac{2M}{x_0 y_0 z_0}$

Step 3: Plug in and integrate.

$$I = \int_0^{x_0} r^2 dm = \int_0^{x_0} x^2 dm = \int_0^{x_0} x^2 \rho z_0 (mx + b)dx = \int_0^{x_0} x^2 \frac{2M}{x_0 y_0 z_0} z_0 (mx + b)dx$$

$$I = \frac{2M}{x_0 y_0}\int_0^{x_0} (mx^3 + bx^2)dx = \frac{2M}{x_0 y_0}\left[\frac{1}{4}mx^4 + \frac{1}{3}bx^3\right]_0^{x_0} = \frac{2M}{x_0 y_0}\left[\frac{1}{4}mx_0^4 + \frac{1}{3}bx_0^3\right]$$

$$I = \frac{2M}{x_0 y_0}\left[\frac{1}{4}\left(-\frac{y_0}{x_0}\right)x_0^4 + \frac{1}{3}(y_0)x_0^3\right] = \frac{2M}{x_0 y_0}\left[-\frac{1}{4}y_0 x_0^3 + \frac{1}{3}y_0 x_0^3\right] = \frac{2M}{x_0 y_0}\left[\frac{1}{12}y_0 x_0^3\right] = \frac{1}{6}Mx_0^2$$

- Most of the time you will simply pull a formula from a table of standard shapes.

Hoop about cylinder axis $I = MR^2$	Solid cylinder (or disk) about cylinder axis $I = \frac{1}{2}MR^2$	Thin rod about axis through center ⊥ to length $I = \frac{1}{12}ML^2$	Annular cylinder (or ring) about cylinder axis $I = \frac{1}{2}M(R_1^2 + R_2^2)$	Slab about ⊥ axis through center $I = \frac{1}{12}M(a^2 + b^2)$
Hoop about any diameter $I = \frac{1}{2}MR^2$	Solid cylinder (or disk) about central diameter $I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$	Thin rod about axis through one end ⊥ to length $I = \frac{1}{3}ML^2$	Solid sphere about any diameter $I = \frac{2}{5}MR^2$	Thin spherical shell about any diameter $I = \frac{2}{3}MR^2$

- Moment of inertia is dependent upon the choice of axis.
- If you have multiple objects with the same axis of rotation you simply add their moments of inertia. (Moments of inertia sum)
- The Parallel Axis Theorem allows you to calculate the moment of inertia of an object rotating around an axis that doesn't pass through its center of mass. To do this you need the moment of inertia for an axis parallel to the axis of rotation and passing through the center of mass (I_{CM}) and the separation of the two axes (h).

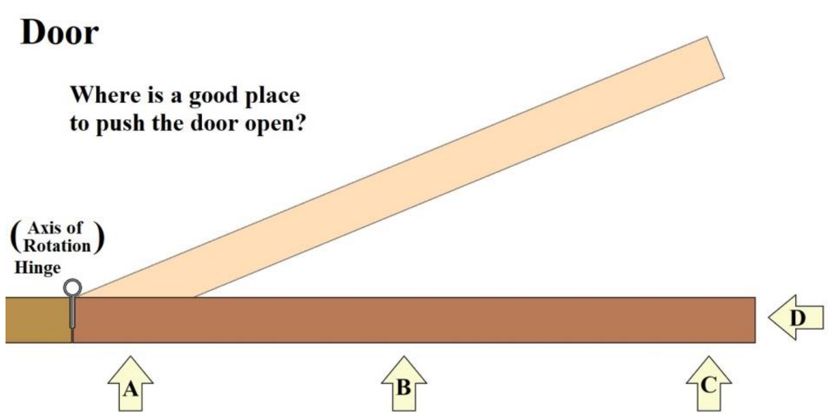
$$I = I_{cm} + Mh^2$$

Example: Find the moment of inertia of a thin rod about axis through one end ⊥ to length via the parallel axis theorem.

$$I = I_{cm} + Mh^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{12}ML^2 + \frac{1}{4}ML^2 = \frac{1}{3}ML^2$$

Torque (τ): $\tau = rF_y = rF\sin\theta \qquad \tau = Fl = Fr\sin\theta \qquad |\vec{\tau}| = |\vec{r} \times \vec{F}| = Fr\sin\theta$

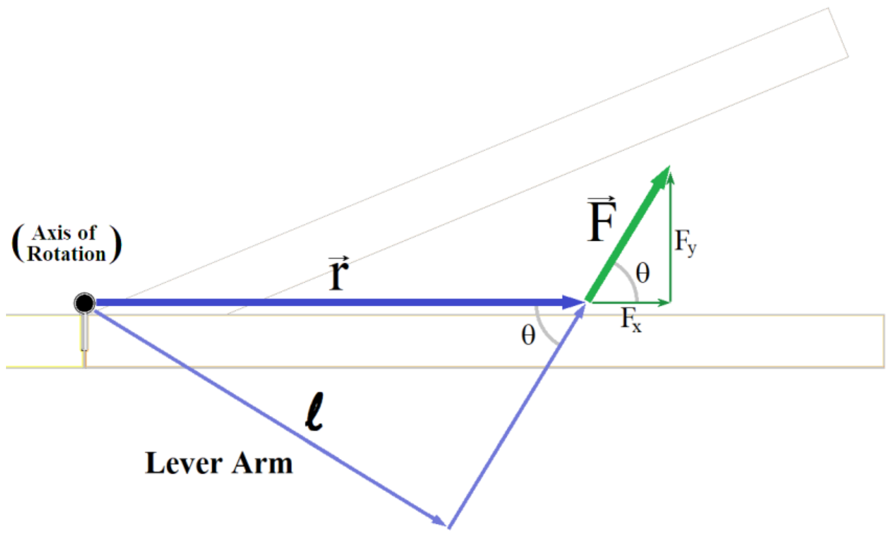
Moment of inertia of most objects is fixed (constant). In those cases, Newton's 2nd law ($\vec{\tau} = I\vec{\alpha}$) indicates torque (τ) and angular acceleration (α) are proportional. This allows us to use the behavior of an object (its angular acceleration) to indicate how much torque was delivered when various forces are applied to it. For our example we will use a door.



Force A doesn't work very well. A lot of force leads to little movement of the door.
Force C is the best option. A little force here is usually enough to open the door.
Force B requires more force than C, but not as much as A.

Force D doesn't open the door at all.

- Any force that acts through the axis of rotation generates no torque.
  - To generate torque a force must have a a component $\perp$ to the line connecting the axis of rotation to the point where the force acts.
- The further from the axis of rotation the force is applied the greater the torque.



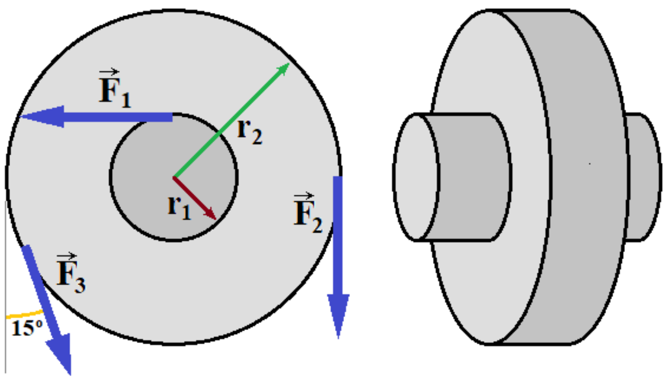
If we apply a force $\vec{F}$ to an object, we define the vector $\vec{r}$ to start at the axis of rotation and at the position where the force is applied. θ is defined to be the angle between $\vec{F}$ and $\vec{r}$.

- Breaking $\vec{F}$ into component (parallel and perpendicular to $\vec{r}$) we find:
  - F_x (the component parallel to $\vec{r}$) generates NO torque.
  - F_y (the component perpendicular to $\vec{r}$) generates positive torque as it rotates the door counter clockwise (CCW).
    - The magnitude of the torque ($\tau = |\vec{\tau}|$) is given by: $\tau = rF_y = rF\sin\theta$
  - Forces that create clockwise (CW) rotations are generating negative torque.
- Breaking $\vec{r}$ into component (parallel and perpendicular to $\vec{F}$) we find:
  - The component parallel to $\vec{F}$ has no bearing on the torque at all.
  - The component perpendicular to $\vec{F}$ is called the <u>Lever Arm</u> (l), and it is directly related to the torque. Any increase in the lever arm gives a proportional increase in torque ($\vec{\tau}$).
    - The magnitude of the torque ($\tau = |\vec{\tau}|$) is given by: $\tau = Fl = Fr\sin\theta$
    - We gain "Leverage" by increasing the lever arm.
- Both viewpoints give the same result, which is often represented as a vector cross product.

$$|\vec{\tau}| = \left|\vec{r} \times \vec{F}\right| = Fr\sin\theta$$

There are more advanced methods of calculating vector cross products, but these are rarely used for torque (as we already know the direction along the axis of rotation).

Example: Three forces act on a compound wheel as shown. The forces come from ropes wrapped around the edges of the wheel. The moment of inertia of the wheel is 30.0 kg·m^2 with an inner radius of r_1= 25.0 cm and an outer radius of r_2 = 50.0 cm. Determine the angular acceleration of the wheel in response to the three forces: F_1 = 80.0 N, F_2 = 30.0 N, and F_3 = 20.0 N.



Use Newton's 2nd Law: $\sum \tau = I\alpha$

To find the torque for each force vector:

(1) Determine sign on torque: CCW is '+', CW is '-'

(2) Find the vector $\vec{r}$*, note it's magnitude*

(3) Determine the angle between $\vec{r}$ *and* $\vec{F}$ *(that's* θ*)*

(4) $\tau = rF\sin\theta$

$$\sum \tau = r_1F_1\sin 90° - r_2F_2\sin 90° + r_2F_3\sin 90° = r_1F_1 - r_2F_2 + r_2F_3$$

$$\sum \tau = (0.250\,m)(80.0N) - (0.500\,m)(30.0\,N) + \big((0.500\,m)\big)(20.0\,N)$$

$$\sum \tau = 20.0\,N\cdot m - 15.0\,N\cdot m + 10.0\,N\cdot m = 15.0\,N\cdot m$$

$$\alpha = \frac{\tau_{Net}}{I} = \frac{15.0\,N\cdot m}{30.0\,kg\cdot m^2} = 0.500\frac{rad}{s^2}$$