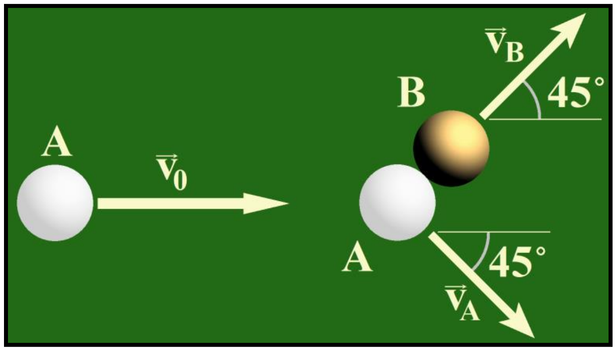
# Lecture 12: Momentum in 2D, Center of Mass, and Rotational Kinematics

*Physics for Engineers & Scientists* (Giancoli): Chapters 9 & 10
*University Physics V1* (Openstax): Chapter 9 & 10

**Example:** The cue ball approaches a stationary ball of equal mass at 3.00 m/s. After the collision the balls separate, the velocity of each ball making a 45.0° angle with the cue ball's original path just on opposite sides. Determine the velocity of both balls after the collision.




Y-Components: $P_{final-y} = P_{init-y}$ $\quad m_B v_B \sin\theta_B - m_A v_A \sin\theta_A = 0$

$$m_B v_B \sin\theta_B = m_A v_A \sin\theta_A \qquad v_B = v_A$$

Y-Components: $P_{final-x} = P_{init-x}$ $\quad m_B v_B \cos\theta_B + m_A v_A \cos\theta_A = m_A v_0$

$$v_B \cos 45° + v_A \cos 45° = v_0 \qquad v_A \cos 45° + v_A \cos 45° = v_0$$

$$2v_A \cos 45° = v_0 \qquad v_A = \frac{v_0}{2\cos 45°} = \frac{3.00\frac{m}{s}}{2\cos 45°} = 2.12132\frac{m}{s} \qquad v_B = v_A = 2.12\frac{m}{s}$$

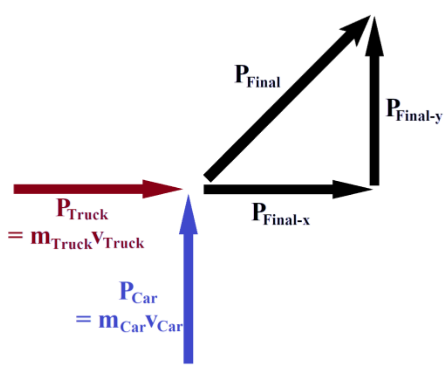
*Is this an elastic or inelastic collision?*

$$E_{init} = \frac{1}{2}mv_0^2 = \frac{1}{2}m\left(3.00\frac{m}{s}\right)^2 = \left(4.50\frac{m^2}{s^2}\right)m$$

$$E_{final} = \frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2 = mv_A^2 = m\left(2.12132\frac{m}{s}\right)^2 = \left(4.50\frac{m^2}{s^2}\right)m$$

*So it's elastic.*

**Example**: A car ($m_{car}$ = 1300 kg) is heading north at 20.0 m/s when it collides with a truck ($m_{truck}$ = 2000 kg) heading east at 15.0 m/s. During the collision the bumpers lock, holding the car and truck together. Determine the velocity of the pair after the collision.




$$P_{Final-x} = m_{Truck}v_{Truck} = (2000\,kg)\left(15.0\frac{m}{s}\right)$$
$$= 30{,}000\,kg\cdot\frac{m}{s}$$

$$P_{Final-y} = m_{car}v_{car} = (1300\,kg)\left(20.0\frac{m}{s}\right)$$
$$= 26{,}000\,kg\cdot\frac{m}{s}$$

$$P_{Final} = \sqrt{P_{Final-x}^2 + P_{Final-y}^2} = \sqrt{\left(30{,}000\ kg \cdot \frac{m}{s}\right)^2 + \left(26{,}000\ kg \cdot \frac{m}{s}\right)^2} = 39{,}699\ kg \cdot \frac{m}{s}$$

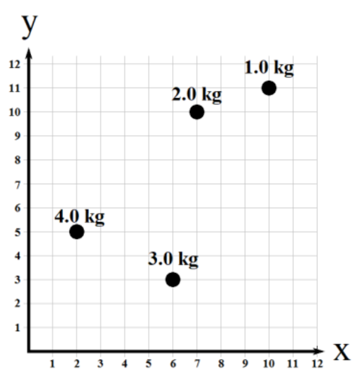
$$v_{Final} = \frac{P_{Final}}{m} = \frac{39{,}699\ kg \cdot \frac{m}{s}}{(2000\ kg + 1300\ kg)} = 12.0 \frac{m}{s}$$

## Center of Mass

- The dynamics of any object are equivalent to having the entire mass at a single point, the center of mass.
  - This allows us to treat every object as a point with mass.
  - As this is also true for gravity, the center of mass is also called the center of gravity.
- The Center of Mass of an object is the mean (average) position of its mass.
  - For scattered point masses:

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \qquad y_{CM} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}$$

**Example**: Find the center of mass.



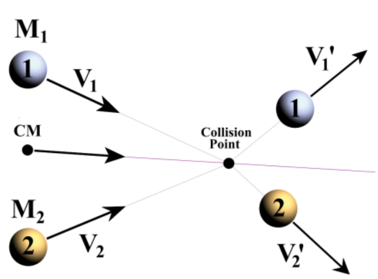

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4}{m_1 + m_2 + m_3 + m_4}$$

$$x_{CM} = \frac{(4.0\ kg)(2.0\ m) + (3.0\ kg)(6.0\ m) + (2.0\ kg)(7.0\ m) + (1.0\ kg)(10.0\ m)}{4.0\ kg + 3.0\ kg + 2.0\ kg + 1.0\ kg} = 5.0\ m$$

$$y_{CM} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + m_4 y_4}{m_1 + m_2 + m_3 + m_4}$$

$$y_{CM} = \frac{(4.0\ kg)(5.0\ m) + (3.0\ kg)(3.0\ m) + (2.0\ kg)(10.0\ m) + (1.0\ kg)(11.0\ m)}{4.0\ kg + 3.0\ kg + 2.0\ kg + 1.0\ kg} = 6.0\ m$$

- The center of mass of a system of particles of net mass M, moves like a particle of net mass M. When subject to the next external forces it accelerates according to $F_{Net} = Ma$.




$$v_{cm} = \frac{dx_{cm}}{dt} = \frac{d}{dt}\left(\frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}\right)$$

$$v_{cm} = \frac{d}{dt}\left(\frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots}{M}\right)$$

$$v_{cm} = \frac{m_1}{M}\frac{dx_1}{dt} + \frac{m_2}{M}\frac{dx_2}{dt} + \frac{m_3}{M}\frac{dx_3}{dt} + \cdots$$

$$v_{cm} = \frac{m_1 v_1}{M} + \frac{m_2 v_2}{M} + \frac{m_3 v_3}{M} + \cdots = \frac{m_1 v_1{}'}{M} + \frac{m_2 v_2{}'}{M} + \frac{m_3 v_3{}'}{M} + \cdots$$

- For mass distributions:

$$x_{CM} = \frac{\int x dm}{\int dm} \qquad y_{CM} = \frac{\int x dm}{\int dm}$$

- The object is broken into infinitesimally small pieces where ‘dm’ is the mass of any given piece. These are then added together (integration).
- Typically, dm is written in terms of volume and density.

$$m = \rho \cdot V \quad \text{therefore...} \quad dm = \rho \cdot dV$$

**Example:** Find the x-component of the center of mass of the triangle shown. Assume uniform thickness and density.

Step 1: Split mass into small pieces, each with the same value of x (as x appears in our equation)

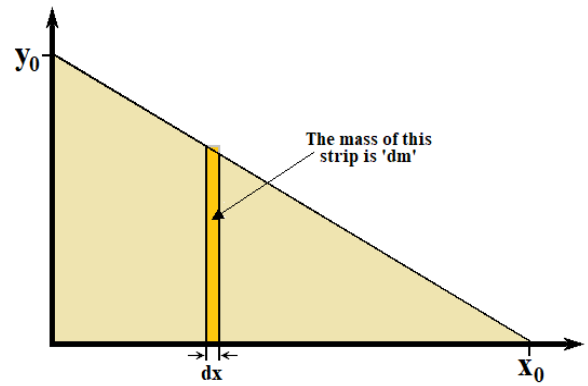
Step 2: Find ‘dm’, the mass of the strip.

$$dm = \rho dV = \rho z_0 A = \rho z_0 y dx$$

*In this instance, we must find y as a function of x (since y varies with x).*

$$y = mx + b \qquad m = \frac{\Delta y}{\Delta x} = \frac{0 - y_0}{x_0 - 0} = -\frac{y_0}{x_0}$$

$$b = y_0 \qquad dm = \rho z_0 y dx = \rho z_0 (mx + b) dx$$




Step 3: Plug in and integrate.

$$x_{CM} = \frac{\int x dm}{\int dm} = \frac{\int_0^{x_0} x[\rho z_0(mx+b)dx]}{\int_0^{x_0} \rho z_0 (mx+b)dx} = \frac{\rho z_0 \int_0^{x_0}(mx^2+bx)dx}{\rho z_0 \int_0^{x_0}(mx+b)dx} = \frac{\int_0^{x_0}(mx^2+bx)dx}{\int_0^{x_0}(mx+b)dx}$$

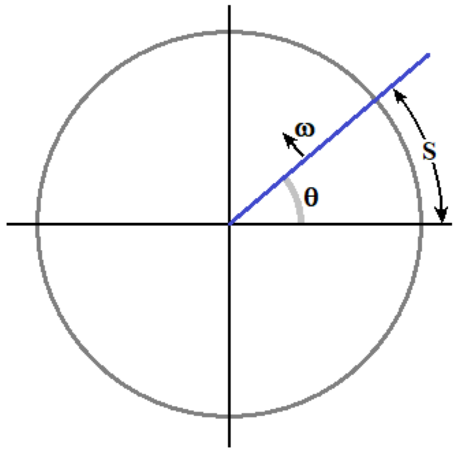
$$x_{CM} = \frac{\left[\frac{1}{3}mx^3 + \frac{1}{2}bx^2\right]_0^{x_0}}{\left[\frac{1}{2}mx^2 + bx\right]_0^{x_0}} = \frac{\frac{1}{3}mx_0^3 + \frac{1}{2}bx_0^2}{\frac{1}{2}mx_0^2 + bx_0} = \frac{2mx_0^3 + 3bx_0^2}{3mx_0^2 + 6bx_0} = \frac{2\left(-\frac{y_0}{x_0}\right)x_0^3 + 3y_0x_0^2}{3\left(-\frac{y_0}{x_0}\right)x_0^2 + 6y_0x_0}$$

$$x_{CM} = \frac{-2y_0x_0^2 + 3y_0x_0^2}{-3y_0x_0 + 6y_0x_0} = \frac{y_0x_0^2}{3y_0x_0} = \frac{1}{3}x_0$$

## Rotational Kinematics

*For rotating objects, velocity is not a universal variable. Different parts of the object move at different speeds. Distance (x) is also not a universal variable. Consequently, these are not good variables to describe rotational motion.*

- Angular Position ($\theta$) fills the role of x.
- Initial Angular Position ($\theta_0$) is the angular position at t=0, and it fills the role of $x_0$.
- Angular Displacement ($\Delta\theta = \theta - \theta_0$) fills the role of $\Delta x$.
- Arc Length (S) is the distance a part of the rigid object moves.




*1 rotation/revolution = 2π radians = 360°*

$S = r \cdot \Delta\theta$ (*radians*)

$\theta = \frac{S}{r} \qquad \frac{meters}{meters} = no\ units$

*Note: 'Radians' is a 'dummy' unit.*

- Angular Velocity (ω) fills the role of v. (*ω is also called 'Angular Frequency'*)
- Initial Angular Velocity ($\omega_0$) is the angular velocity at t=0, and it fills the role of $v_0$.

*Note: 'ω' is a lower-cased Greek letter omega. Not W.*

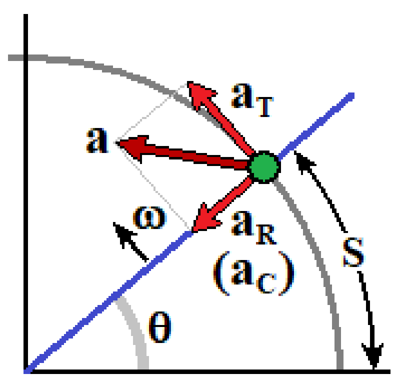
$$\omega_{Avg} = \frac{\Delta\theta}{\Delta t} = \frac{\theta - \theta_0}{t} \qquad \omega = \frac{d\theta}{dt} \qquad \text{Units: } \frac{rad}{s} \qquad r\omega = r\frac{d\theta}{dt} = \frac{d(r\theta)}{dt} = \frac{dS}{dt} = v$$

- The Period (T) is the time needed to make one full revolution.
- The Frequency (f) is the rotation rate (number of revolutions per unit time)

$$\text{Units: } 1\ \frac{\text{Revolution}}{\text{Second}} = 1\ \text{s}^{-1} = 1\ \text{Hz} \qquad 60\ \text{RPM} = 60\ \frac{\text{Revolution}}{\text{Minute}} = 1\ \text{Hz}$$

$$f = \frac{1}{T} \qquad \omega = 2\pi f = \frac{2\pi}{T} \qquad v = \frac{2\pi r}{T} = \omega r$$

- In rotational motion, Acceleration (a) gets broken down into two components.




- Tangential Accleration ($a_T$) changes the speed (magnitude of velocity) of an object moving in a circle.
- Radial Accleration ($a_R$), equivalent to centripetal acceleration ($a_C$), changes the direction but not the speed of an object moving in a circle.

$$a = \sqrt{a_T^2 + a_R^2}$$

- Angular Acceleration (α) fills the role of a.

$$\alpha_{Avg} = \frac{\Delta\omega}{\Delta t} = \frac{\omega - \omega_0}{t} \qquad \alpha = \frac{d\omega}{dt} \qquad \text{Units: } \frac{rad}{s^2}$$

$$r\alpha = r\frac{d\omega}{dt} = \frac{d(r\omega)}{dt} = \frac{dv}{dt} = a_T \qquad a_R = a_C = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2$$

- For the special case of constant angular acceleration, a set of equations can be found from the one-dimensional kinematic equations for constant acceleration.

$$v = v_0 + a_T t \qquad \omega r = \omega_0 r + \alpha r t \qquad \omega = \omega_0 + \alpha t$$

$$s = s_0 + \frac{1}{2}(v + v_0)t \qquad r\theta = r\theta_0 + \frac{1}{2}(r\omega + r\omega_0)t \qquad \theta = \theta_0 + \frac{1}{2}(\omega + \omega_0)t$$

$$s = s_0 + v_0 t + \frac{1}{2}a_T t^2 \qquad r\theta = r\theta_0 + r\omega_0 t + \frac{1}{2}r\alpha t^2 \qquad \theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$v^2 = v_0^2 + 2a_T(s - s_0) \qquad r^2\omega^2 = r^2\omega_0^2 + 2r\alpha(r\theta - r\theta_0) \qquad \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

  - Rotational kinematics are similar to one-dimensional kinematics (just a change of variable) and solved the same way.
  - There are four variables (θ, ω, α, and t) and two constants ($\theta_0$ and $\omega_0$).
  - Three of the variables are related in each of the four equations. In many cases you can find the equation you need by determining which variable is absent.

**Example:** A wind turbine is activated as the winds reach a threshold. The blades start from rest and accelerate uniformly to an angular velocity of 9.87 rpms in 27.4 s. Determine the angular acceleration of the blades.

Extract Data: $\omega_0 = 0$ ω = 9.87 rpms = 1.03358 rad/s t = 27.4 s α = ???

$$\omega = 9.87\left(\frac{Rev}{Min}\right)\left(\frac{1\ Min}{60\ s}\right)\left(\frac{2\pi\ rad}{1\ Rev}\right) = 1.03358\frac{rad}{s}$$

*Be warned: 2π/60 = 0.10472. If you fail to do this conversion, you will be off by a factor that is close to a power of 10.*

No information about position is given. The equation without position is…

$$\omega = \omega_0 + \alpha t = \alpha t \qquad \alpha = \frac{\omega}{t} = \frac{1.03358\frac{rad}{s}}{27.4\ s} = 0.03772\frac{rad}{s^2}$$

**Example:** A grinding wheel undergoes uniform angular acceleration from rest to 680 rad/s over 1.30 seconds. Then the power is removed and friction causes it to decelerate back to rest in 18.7 seconds. Through what angle does the wheel turn during this time?

*There are two different accelerations (both constants). This requires two sets of equations, one for the acceleration and one for the deceleration. This is an odd case where both are the same.*

Accelerating:

Extract Data: $\theta_0 = 0$ $\theta = ???$ $\omega_0 = 0$ $\omega = 680\frac{rad}{s}$ $\alpha =$ $t = 1.30\ s$

Equation with no α: $\theta = \theta_0 + \frac{1}{2}(\omega + \omega_0)t$ $\theta = \frac{1}{2}\omega t = \frac{1}{2}\left(680\frac{rad}{s}\right)(1.30\ s) = 442\ rad$

Decelerating:

Extract Data: $\theta_0 = 442\ rad$ $\theta = ???$ $\omega_0 = 680\frac{rad}{s}$ $\omega = 0$ $\alpha =$ $t = 18.7\ s$

Equation with no α: $\theta = \theta_0 + \frac{1}{2}(\omega + \omega_0)t$

$$\theta = \theta_0 + \frac{1}{2}\omega_0 t = 442\,rad + \frac{1}{2}\left(680\frac{rad}{s}\right)(18.7\,s) = 6800\,rad$$

*Alternatively, one could note that the average angular velocity is the same value whether accelerating or decelerating.*

*The solution would just be $\omega_{avg}t_{net}$*

$$\omega_{avg} = \frac{1}{2}\omega = \frac{1}{2}\left(680\frac{rad}{s}\right) = 340\frac{rad}{s} \qquad \theta = \omega_{avg}t = \left(340\frac{rad}{s}\right)(1.30\,s + 18.7\,s) = 6800\,rad$$