Lecture 10: Potential Energy and Conservation of Energy

Physics for Engineers & Scientists (Giancoli): Chapter 8
University Physics V1 (Openstax): Chapter 8

Gravitational Potential Energy $U_G = mgh$

- When an object moves against the force of gravity, the work done lifting it becomes stored in the objects position (gravitational potential energy).
- When that object is released, the entirety of the stored energy is converted back into kinetic energy as it falls (assuming friction/wind resistance is not present).
- By definition, the potential energy of a force (U) differs by a negative sign from the work done moving against that force.

$$U = -W$$

- On the surface of the Earth, the gravitational force is effectively constant and equal to an object's weight (F = mg). Moving upward, the distance it moves would be the change in height. The force of gravity is directed opposite to the motion.

$$U_g = -W_g = -F_g \cdot d \cdot Cos(\theta) = -mg \cdot h \cdot Cos(180°) = mgh$$

- As we can choose to place our coordinate axis anywhere we desire, the gravitational potential energy of an object may vary with our choice of origin. Consequently, gravitational potential energy doesn't have an absolute value. Only the change in potential energy is relevant.

Conservation of Energy

- Energy is neither created nor destroyed. It only changes from one form to another.
- If work is done to an object due to external forces, it must be converted into either kinetic energy or potential energy (or some of both).

$$W_{NC} = \Delta KE + \Delta U$$

$$\Delta KE = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 \qquad \Delta U = mgy - mgy_0$$

  - The work done by non-conservative forces (W_{NC}) includes any energy added by applied forces as well as energy lost due to friction. Gravity is excluded as it is a conservative force and is covered under potential energy.
  - Conservative forces are those where the work done moving from one point to another does not depend on the path. Any work done by these forces is stored as potential energy (which might be returned later).

- It can be useful to rearrange this equation to find: $\boldsymbol{E_{init} + E_{added} = E_{final}}$

$$W_{NC} = \Delta KE + \Delta U = (KE_{final} - KE_{init}) + (U_{final} - U_{init})$$

$$W_{NC} = (KE_{final} + U_{final}) - (KE_{init} + U_{init}) = E_{final} - E_{init}$$

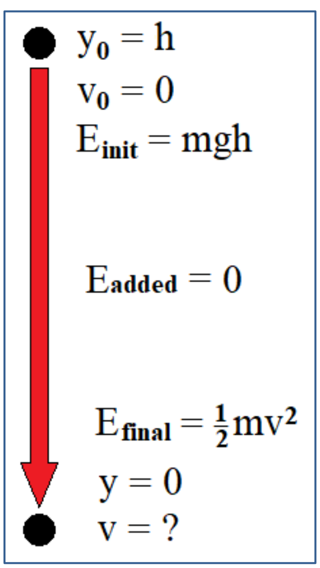
$$E_{init} + W_{NC} = E_{final} \qquad E_{init} + E_{added} = E_{final}$$

- The variable time does not show up directly in any form of energy. Any problem that includes motion without a collision and makes no mention of time is a good candidate to solve using conservation of energy.
- Energy lost due to friction is not actually lost, but rather it is transformed in heat (molecular vibrations).

Solving Problems with Conservation of Energy

- You only need to consider the initial and final states. Intermediate states are irrelevant.
- When you determine the initial and final energies, make sure to include every form of energy present.
  - Anything in motion will have kinetic energy.
  - Anything not at ground level (or where you decided to place y=0) will have gravitational potential energy.
- Make sure to traverse the path in between the initial and final positions to include anything that adds or removes energy to find E_{added} (W_{NC}).
  - Any friction forces will remove energy.
  - Other applied forces may add or subtract energy.
  - Gravity is accounted for with potential energy (don't include that with Eadded).

Example: A woman drops a small rock off a balcony 10.2 m above the ground. Assuming wind resistance is negligible, how fast is the rock moving just before it hits the ground?



We will set y = 0 to be at ground level.

Note: Setting y = 0 to occur at ground level is typically the most 'comfortable' thing to do. However, you may find that matching y = 0 to your lowest object is preferable. In this problem, both give the same origin.

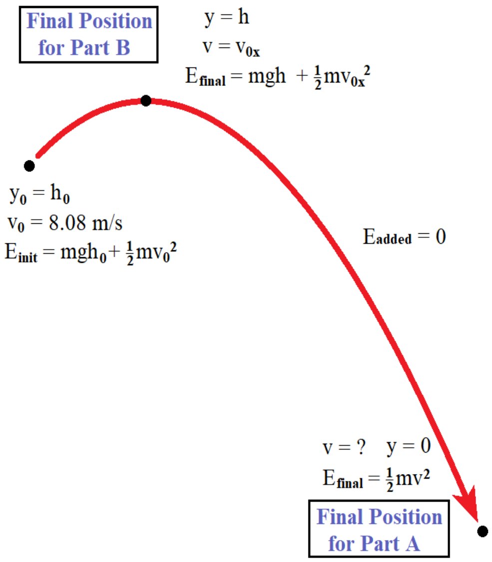
$$E_{init} = mgh \qquad E_{final} = \frac{1}{2}mv^2 \qquad E_{added} = 0$$

$$E_{init} + E_{added} = E_{final} \qquad mgh = \frac{1}{2}mv^2$$

$$gh = \frac{1}{2}v^2 \qquad 2gh = v^2$$

$$v = \sqrt{2gh} = \sqrt{2\left(9.80\frac{m}{s^2}\right)(10.2\ m)} = \sqrt{199.92\frac{m^2}{s^2}} = 14.1\frac{m}{s}$$

Example: A woman throws a small rock off a balcony 10.2 m above the ground. The initial velocity is 8.08 m/s and directed 30.0° above the horizon. Assume wind resistance is negligible. Determine A) the speed of the rock just before it hits the ground, and B) the maximum height of the rock.



We will set y = 0 to be at ground level.

Part A:

$$E_{init} = mgh + \frac{1}{2}mv_0^2 \qquad E_{final} = \frac{1}{2}mv^2 \qquad E_{added} = 0$$

$$E_{init} + E_{added} = E_{final} \qquad mgh + \frac{1}{2}mv_0^2 = \frac{1}{2}mv^2$$

$$gh + \frac{1}{2}v_0^2 = \frac{1}{2}v^2 \qquad 2gh + v_0^2 = v^2$$

$$v = \sqrt{2gh + v_0^2} = \sqrt{2\left(9.80\frac{m}{s^2}\right)(10.2\ m) + \left(8.08\frac{m}{s}\right)^2}$$

$$v = \sqrt{265.2064\frac{m^2}{s^2}} = 16.3\frac{m}{s}$$

Part B: $E_{init} = mgh_0 + \frac{1}{2}mv_0^2 \qquad E_{final} = mgh + \frac{1}{2}mv^2 \qquad E_{added} = 0$

$$v_y = 0 \qquad v = v_x = v_{0x} = v_0 \cdot Cos(\theta) = \left(8.08\frac{m}{s}\right)\cos(30.0°) = 6.9975\frac{m}{s}$$

$$E_{init} + E_{added} = E_{final} \qquad mgh_0 + \frac{1}{2}mv_0^2 = mgh + \frac{1}{2}mv^2 \qquad gh_0 + \frac{1}{2}v_0^2 = gh + \frac{1}{2}v^2$$

$$gh_0 + \frac{1}{2}v_0^2 - \frac{1}{2}v^2 = gh \qquad h_0 + \frac{v_0^2}{2g} - \frac{v^2}{2g} = h \qquad h = h_0 + \frac{v_0^2 - v^2}{2g}$$

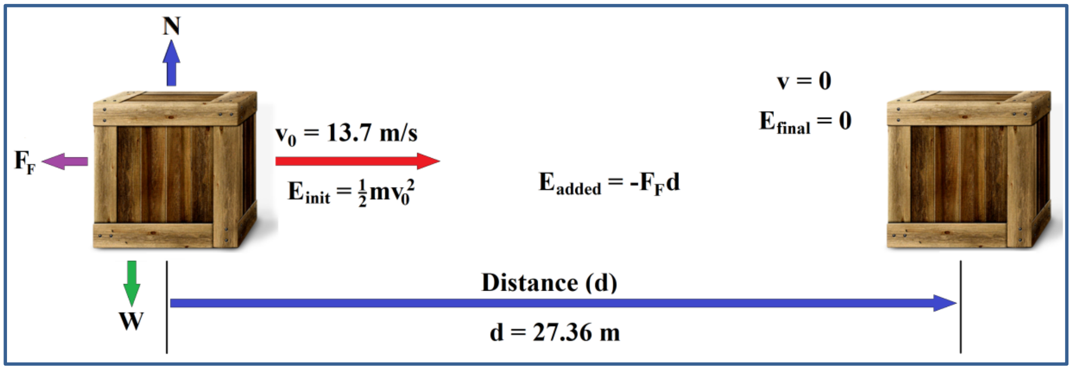
$$h = h_0 + \frac{v_0^2 - v^2}{2g} = 10.2\ m + \frac{\left(8.08\frac{m}{s}\right)^2 - \left(6.9975\frac{m}{s}\right)^2}{2\left(9.80\frac{m}{s^2}\right)} = 11.0\ m$$

Alternatively: $v_0^2 - v^2 = \left(v_{0x}^2 + v_{0y}^2\right) - v_{0x}^2 = v_{0y}^2$

$$v_{0y} = v_0 \cdot \sin(\theta) = \left(8.08\frac{m}{s}\right) \cdot \sin(30.0°) = 4.04\frac{m}{s}$$

$$h = h_0 + \frac{v_0^2 - v^2}{2g} = h_0 + \frac{v_{0y}^2}{2g} = 10.2\ m + \frac{\left(4.04\frac{m}{s}\right)^2}{2\left(9.80\frac{m}{s^2}\right)} = 11.0\ m$$

Example: Initially a crate is sliding on a horizontal surface at 13.7 m/s. The crate moves a distance of 27.36 m before coming to rest. Determine the coefficient of kinetic friction between the crate and the surface.



$$E_{init} = \frac{1}{2}mv_0^2 \qquad E_{final} = 0 \qquad E_{added} = -F_F d = -\mu_k N d = -\mu_k mgd$$

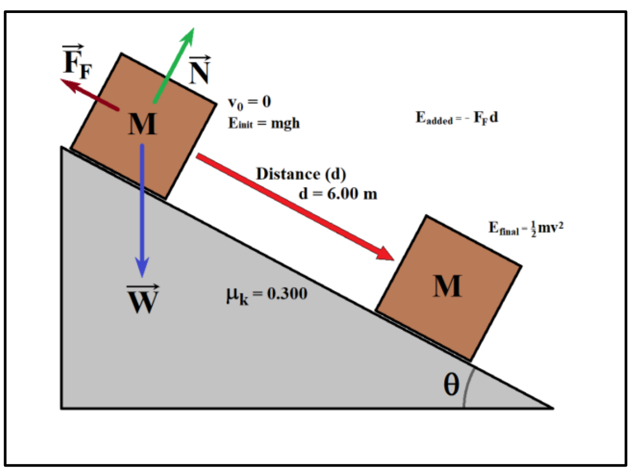
Note: As there are only two vertical forces (N and W) that must cancel, N=W=mg

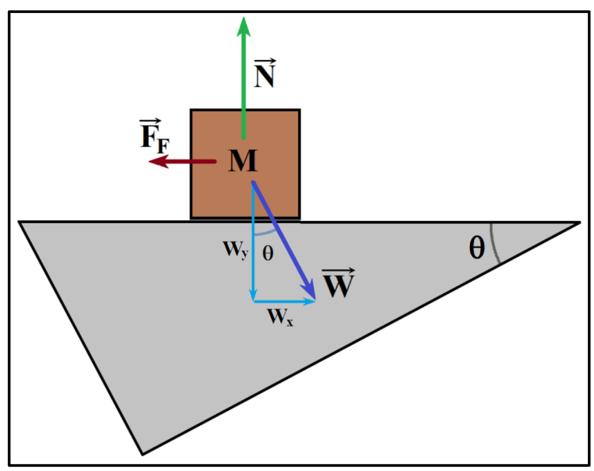
$$E_{init} + E_{added} = E_{final} \qquad \frac{1}{2}mv_0^2 - \mu_k mgd = 0 \qquad \frac{1}{2}mv_0^2 = \mu_k mgd \qquad \frac{1}{2}v_0^2 = \mu_k gd$$

$$\mu_k = \frac{v_0^2}{2gd} = \frac{\left(13.7\frac{m}{s}\right)^2}{2\left(9.80\frac{m}{s^2}\right)(27.36\,m)} = 0.350$$

Example: Initially a crate is at rest at the top of a 30.0° incline. The coefficient of kinetic friction between the crate and the surface is 0.300. How fast is the crate moving after sliding 6.00 m down the incline?

We will set y = 0 to be at ground level.





$$E_{init} = mgh = mgd \cdot \sin(30.0°) = \frac{1}{2}mgd \qquad E_{final} = \frac{1}{2}mv^2$$

The weight (W) does work, but this is not included in E_{added} as it is already covered with potential energy.

The normal force (N) doesn't do any work as it is perpendicular to the direction of motion.

$$E_{added} = -F_F d = -\mu_k N d = -\mu_k W_y d = -\mu_k mgd\cos\theta$$

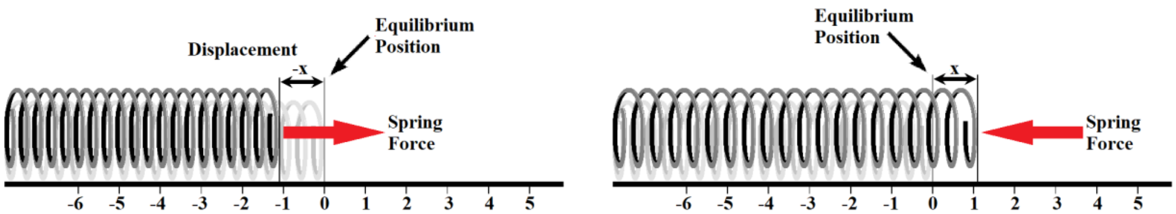
$$E_{init} + E_{added} = E_{final} \qquad \tfrac{1}{2}mgd - \mu_k mgd\cos\theta = \tfrac{1}{2}mv^2$$

$$mgd - 2\mu_k mgd\cos\theta = mv^2 \qquad gd - 2\mu_k gd\cos\theta = v^2 \qquad v^2 = gd(1 - 2\mu_k\cos\theta)$$

$$v = \sqrt{gd(1 - 2\mu_k\cos\theta)} = \sqrt{\left(9.80\frac{m}{s^2}\right)(6.00\,m)\{1 - 2(0.300)\cos(30.0°)\}} = 5.31\frac{m}{s}$$

Springs and Hooke's Law

- Springs will naturally return to their initial (equilibrium) position after being stretched or compressed. A force that does this can be referred to as a **Restoring Force**.
- There are limits to how far springs can be stretched or compressed before losing their ability to stretch and becoming permanently deformed. This is called the **Elastic Limit** of the spring.
- With springs it is typical to use x as the change in length from equilibrium (equal to the displacement of the end) with it stretching into the positive axis and compressing into the negative axis.



- Hooke's Law: $F_x = -kx$
  - The negative sign indicates that the force points opposite the direction of displacement.
  - The spring constant, k, is only valid for a specific spring (not a universal constant).
  - Hooke's Law is only a first order linear approximation. Many springs will deviate from Hooke's Law before reaching the elastic limit.
  - Solid surfaces typically obey Hooke's law (albeit with very large spring constants). A very slight compression creates a large restoring force. This allows the normal force to take whatever value it needs to be.
  - Hooke's Law also applies to other objects that behave elastically.
- Elastic Potential Energy: $U_{sp} = \frac{1}{2}kx^2$
  - $U_{sp} = -W_{sp} = -\int_0^x F_{sp}(x)dx = -\int_0^x(-kx)dx = k\int_0^x(x)dx = k\left\{\frac{1}{2}x^2\right\}_0^x = \frac{1}{2}kx^2$
  - When using conservation of energy, it is preferable to include the elastic potential energy of springs rather than include it as work from an applied force (as you will just have to do this integral again).

Example: An archer pulls the bowstring back 42.0 cm and fires a 65.0 g arrow at 57.5 m/s. Determine the maximum height the arrow can reach if the archer pulls the bowstring back 50.0 cm.

For objects that obey Hooke's Law, the dynamic characteristics of that object have been reduced to a single quantity, the spring constant. If you don't know the spring constant of the object, you'll need to find that first.

$$E_{init} = \frac{1}{2}kx_1^2 \qquad E_{final} = \frac{1}{2}mv^2 \qquad E_{added} = 0$$

$$E_{init} + E_{added} = E_{final} \qquad \frac{1}{2}kx_1^2 = \frac{1}{2}mv^2 \qquad kx_1^2 = mv^2$$

$$k = \frac{mv^2}{x_1^2} = \frac{(0.0650\ kg)\left(57.5\frac{m}{s}\right)^2}{(0.420\ m)^2} = 1218.3\frac{N}{m}$$

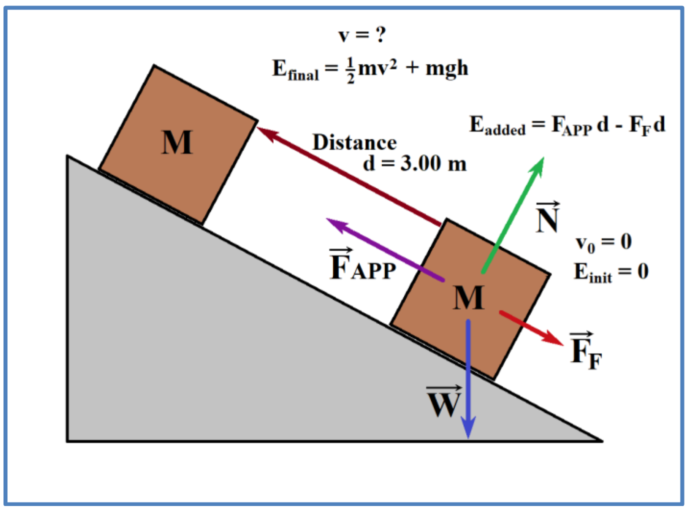
Now that we have the spring constant, we can look for the height.

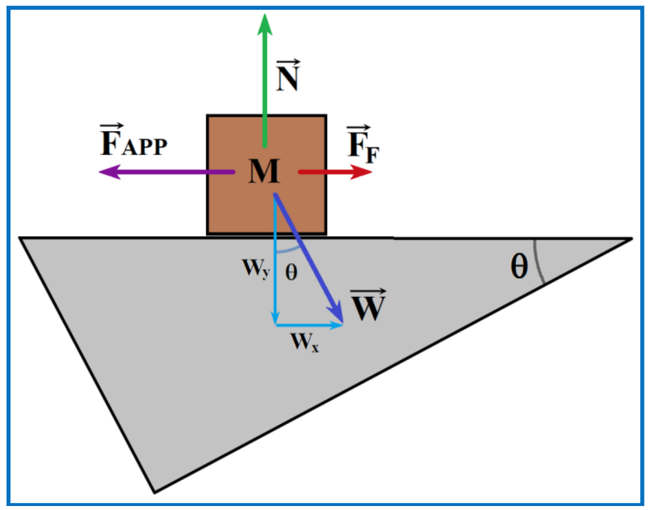
$$E_{init} = \frac{1}{2}kx_2^2 \qquad E_{final} = mgh \qquad E_{added} = 0$$

$$E_{init} + E_{added} = E_{final} \qquad \frac{1}{2}kx_2^2 = mgh$$

$$h = \frac{kx_2^2}{2mg} = \frac{\left(1218.3\frac{N}{m}\right)(0.500\ m)^2}{2(0.0650\ kg)\left(9.80\frac{m}{s^2}\right)} = 239\ m$$

Example: A worker pushes a crate up a 30.0° incline by delivering a 10.0 N force. The coefficient of kinetic friction between the crate and incline is 0.200. If the crate starts from rest and weighs 10.0 N, how fast is the crate moving after it traverses a distance of 3.00 m along the incline?





$$E_{init} = 0 \qquad E_{final} = \frac{1}{2}mv^2 + mgh$$

The weight (W) does work, but this is not included in E_{added} as it is already covered with potential energy.

The normal force (N) doesn't do any work as it is perpendicular to the direction of motion.

$$E_{added} = F_{App}d - F_F d \qquad F_F d = \mu_k N d = \mu_k W_y d = \mu_k mgd\cos\theta$$

$$E_{init} + E_{added} = E_{final} \qquad F_{App}d - \mu_k mgd\cos\theta = \frac{1}{2}mv^2 + mgh$$

$$F_{App}d - \mu_k mgd\cos\theta = \frac{1}{2}mv^2 + mgd\sin\theta \qquad F_{App}d - \mu_k mgd\cos\theta - mgd\sin\theta = \frac{1}{2}mv^2$$

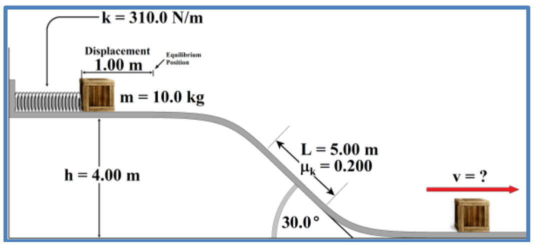
$$2F_{App}d - 2\mu_k mgd\cos\theta - 2mgd\sin\theta = mv^2$$

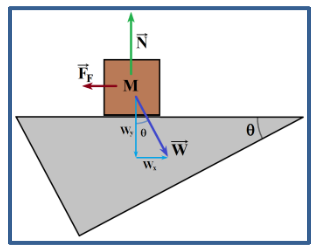
$$\frac{2F_{App}d}{m} - 2\mu_k gd\cos\theta - 2gd\sin\theta = v^2 \qquad \frac{2F_{App}gd}{W} - 2\mu_k gd\cos\theta - 2gd\sin\theta = v^2$$

$$2gd\left(\frac{F_{App}}{W} - \mu_k\cos\theta - \sin\theta\right) = v^2 \qquad v = \sqrt{2gd\left(\frac{F_{App}}{W} - \mu_k\cos\theta - \sin\theta\right)}$$

$$v = \sqrt{2\left(9.80\frac{m}{s^2}\right)(3.00\,m)\left(\frac{10.0\,N}{10.0\,N} - (0.200)\cos(30.0°) - \sin(30.0°)\right)} = 4.38\frac{m}{s}$$

Example: A distribution center has an interesting system for moving crates. A mechanism compresses a spring (k = 310.0 N/m) by 1.00 m. A 10.0 kg crate is then placed in front of the spring, which is then triggered sending the crate on its way. It falls a height of 4.00 m as it moves down a 30.0° incline before leveling off. The entire surface is frictionless except for a 5.00 m long stretch down the incline where the coefficient of friction is 0.200. How fast is the crate moving at the bottom of the incline?





$$E_{init} = \frac{1}{2}kx^2 + mgh \qquad E_{final} = \frac{1}{2}mv^2$$

$$E_{added} = -F_F d = -\mu_k Nd = -\mu_k W_y d = -\mu_k mgd\cos\theta$$

$$E_{init} + E_{added} = E_{final} \qquad \frac{1}{2}kx^2 + mgh - \mu_k mgd\cos\theta = \frac{1}{2}mv^2$$

$$kx^2 + 2mgh - 2\mu_k mgd\cos\theta = mv^2 \qquad \frac{k}{m}x^2 + 2gh - 2\mu_k gd\cos\theta = v^2$$

$$v = \sqrt{\frac{k}{m}x^2 + 2gh - 2\mu_k gd\cos\theta}$$

$$v = \sqrt{\frac{310\frac{N}{m}}{10.0\,kg}(1.00m)^2 + 2\left(9.80\frac{m}{s^2}\right)(4.00\,m) - 2(0.200)\left(9.80\frac{m}{s^2}\right)(5.00\,m)\cos(30°)} = 9.61\text{ m/s}$$