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10.1 Angular Acceleration

Uniform Circular Motion and Gravitation discussed only uniform circular motion, which is motion in a circle at constant speed and, hence, constant angular velocity. Recall that angular velocity ω was defined as the time rate of change of angle θ:

$$\omega = \frac{\Delta\theta}{\Delta t}, \tag{10.1}$$

where θ is the angle of rotation as seen in Figure 10.3. The relationship between angular velocity ω and linear velocity v was also defined in Rotation Angle and Angular Velocity as

$$v = r\omega \tag{10.2}$$

or

$$\omega = \frac{v}{r}, \tag{10.3}$$

where r is the radius of curvature, also seen in Figure 10.3. According to the sign convention, the counter clockwise direction is considered as positive direction and clockwise direction as negative

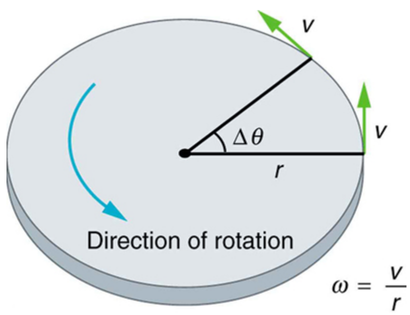

Figure 10.3 This figure shows uniform circular motion and some of its defined quantities.

Angular velocity is not constant when a skater pulls in her arms, when a child starts up a merry-go-round from rest, or when a computer's hard disk slows to a halt when switched off. In all these cases, there is an **angular acceleration**, in which ω changes. The faster the change occurs, the greater the angular acceleration. Angular acceleration α is defined as the rate of change of angular velocity. In equation form, angular acceleration is expressed as follows:

$$\alpha = \frac{\Delta\omega}{\Delta t}, \tag{10.4}$$

where $\Delta\omega$ is the **change in angular velocity** and Δt is the change in time. The units of angular acceleration are $(\text{rad/s})/\text{s}$, or rad/s^2. If ω increases, then α is positive. If ω decreases, then α is negative.

EXAMPLE 10.1

Calculating the Angular Acceleration and Deceleration of a Bike Wheel

Suppose a teenager puts her bicycle on its back and starts the rear wheel spinning from rest to a final angular velocity of 250 rpm in 5.00 s. (a) Calculate the angular acceleration in rad/s^2. (b) If she now slams on the brakes, causing an angular acceleration of $-87.3\ \text{rad/s}^2$, how long does it take the wheel to stop?

Strategy for (a)

The angular acceleration can be found directly from its definition in $\alpha = \frac{\Delta\omega}{\Delta t}$ because the final angular velocity and time are given. We see that $\Delta\omega$ is 250 rpm and Δt is 5.00 s.

Solution for (a)

Entering known information into the definition of angular acceleration, we get

$$\begin{aligned}\alpha &= \frac{\Delta\omega}{\Delta t} \\ &= \frac{250\ \text{rpm}}{5.00\ \text{s}}.\end{aligned} \tag{10.5}$$

Because $\Delta\omega$ is in revolutions per minute (rpm) and we want the standard units of rad/s^2 for angular acceleration, we need to convert $\Delta\omega$ from rpm to rad/s:

$$\begin{aligned}\Delta\omega &= 250\frac{\text{rev}}{\text{min}}\cdot\frac{2\pi\ \text{rad}}{\text{rev}}\cdot\frac{1\ \text{min}}{60\ \text{s}} \\ &= 26.2\frac{\text{rad}}{\text{s}}.\end{aligned} \tag{10.6}$$

Entering this quantity into the expression for α, we get

$$\begin{aligned}\alpha &= \frac{\Delta\omega}{\Delta t} \\ &= \frac{26.2\ \text{rad/s}}{5.00\ \text{s}} \\ &= 5.24\ \text{rad/s}^2.\end{aligned} \tag{10.7}$$

Strategy for (b)

In this part, we know the angular acceleration and the initial angular velocity. We can find the stoppage time by using the definition of angular acceleration and solving for Δt, yielding

$$\Delta t = \frac{\Delta\omega}{\alpha}. \tag{10.8}$$

Solution for (b)

Here the angular velocity decreases from $26.2\ \text{rad/s}$ (250 rpm) to zero, so that $\Delta\omega$ is $-26.2\ \text{rad/s}$, and α is given to be $-87.3\ \text{rad/s}^2$. Thus,

$$\begin{aligned}\Delta t &= \frac{-26.2\ \text{rad/s}}{-87.3\ \text{rad/s}^2} \\ &= 0.300\ \text{s}.\end{aligned} \tag{10.9}$$

Discussion

Note that the angular acceleration as the girl spins the wheel is small and positive; it takes 5 s to produce an appreciable angular velocity. When she hits the brake, the angular acceleration is large and negative. The angular velocity quickly goes to zero. In both cases, the relationships are analogous to what happens with linear motion. For example, there is a large deceleration when you crash into a brick wall—the velocity change is large in a short time interval.

If the bicycle in the preceding example had been on its wheels instead of upside-down, it would first have accelerated along the ground and then come to a stop. This connection between circular motion and linear motion needs to be explored. For example, it would be useful to know how linear and angular acceleration are related. In circular motion, linear acceleration is *tangent* to the circle at the point of interest, as seen in Figure 10.4. Thus, linear acceleration is called **tangential acceleration** a_t.

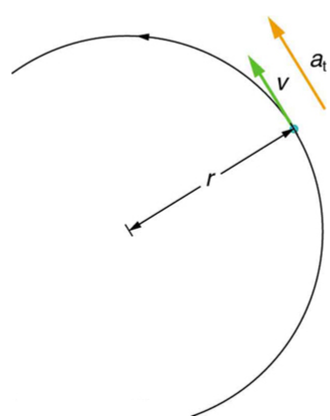

Figure 10.4 In circular motion, linear acceleration a, occurs as the magnitude of the velocity changes: a is tangent to the motion. In the

context of circular motion, linear acceleration is also called tangential acceleration a_t.

Linear or tangential acceleration refers to changes in the magnitude of velocity but not its direction. We know from Uniform Circular Motion and Gravitation that in circular motion centripetal acceleration, a_c, refers to changes in the direction of the velocity but not its magnitude. An object undergoing circular motion experiences centripetal acceleration, as seen in Figure 10.5. Thus, a_t and a_c are perpendicular and independent of one another. Tangential acceleration a_t is directly related to the angular acceleration α and is linked to an increase or decrease in the velocity, but not its direction.

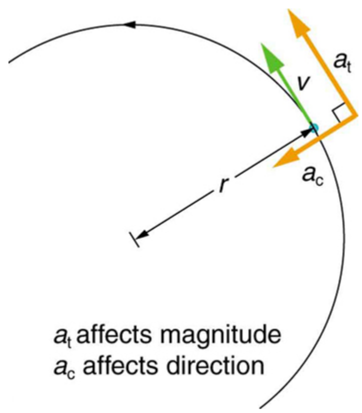

Figure 10.5 Centripetal acceleration a_c occurs as the direction of velocity changes; it is perpendicular to the circular motion. Centripetal and tangential acceleration are thus perpendicular to each other.

Now we can find the exact relationship between linear acceleration a_t and angular acceleration α. Because linear acceleration is proportional to a change in the magnitude of the velocity, it is defined (as it was in One-Dimensional Kinematics) to be

$$a_t = \frac{\Delta v}{\Delta t}. \tag{10.10}$$

For circular motion, note that $v = r\omega$, so that

$$a_t = \frac{\Delta (r\omega)}{\Delta t}. \tag{10.11}$$

The radius r is constant for circular motion, and so $\Delta(r\omega) = r(\Delta\omega)$. Thus,

$$a_t = r\frac{\Delta\omega}{\Delta t}. \tag{10.12}$$

By definition, $\alpha = \frac{\Delta\omega}{\Delta t}$. Thus,

$$a_t = r\alpha, \tag{10.13}$$

or

$$\alpha = \frac{a_t}{r}. \tag{10.14}$$

These equations mean that linear acceleration and angular acceleration are directly proportional. The greater the angular acceleration is, the larger the linear (tangential) acceleration is, and vice versa. For example, the greater the angular acceleration of a car's drive wheels, the greater the acceleration of the car. The radius also matters. For example, the smaller a wheel, the smaller its linear acceleration for a given angular acceleration α.

EXAMPLE 10.2

Calculating the Angular Acceleration of a Motorcycle Wheel

A powerful motorcycle can accelerate from 0 to 30.0 m/s (about 108 km/h) in 4.20 s. What is the angular acceleration of its 0.320-m-radius wheels? (See Figure 10.6.)

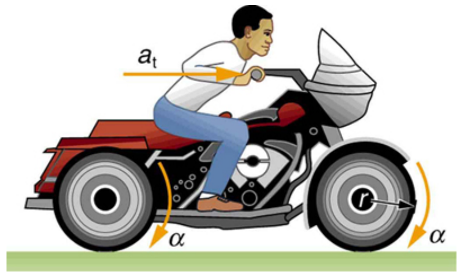

Figure 10.6 The linear acceleration of a motorcycle is accompanied by an angular acceleration of its wheels.

Strategy

We are given information about the linear velocities of the motorcycle. Thus, we can find its linear acceleration a_t. Then, the expression $\alpha = \frac{a_t}{r}$ can be used to find the angular acceleration.

Solution

The linear acceleration is

$$\begin{aligned} a_t &= \frac{\Delta v}{\Delta t} \\ &= \frac{30.0 \text{ m/s}}{4.20 \text{ s}} \\ &= 7.14 \text{ m/s}^2. \end{aligned} \tag{10.15}$$

We also know the radius of the wheels. Entering the values for a_t and r into $\alpha = \frac{a_t}{r}$, we get

$$\begin{aligned} \alpha &= \frac{a_t}{r} \\ &= \frac{7.14 \text{ m/s}^2}{0.320 \text{ m}} \\ &= 22.3 \text{ rad/s}^2. \end{aligned} \tag{10.16}$$

Discussion

Units of radians are dimensionless and appear in any relationship between angular and linear quantities.

So far, we have defined three rotational quantities—θ, ω, and α. These quantities are analogous to the translational quantities x, v, and a. Table 10.1 displays rotational quantities, the analogous translational quantities, and the relationships between them.

Rotational	Translational	Relationship
θ	x	$\theta = \frac{x}{r}$
ω	v	$\omega = \frac{v}{r}$
α	a	$\alpha = \frac{a_t}{r}$

Table 10.1 Rotational and Translational Quantities

Making Connections: Take-Home Experiment

Sit down with your feet on the ground on a chair that rotates. Lift one of your legs such that it is unbent (straightened out). Using the other leg, begin to rotate yourself by pushing on the ground. Stop using your leg to push the ground but allow the chair to rotate. From the origin where you began, sketch the angle, angular velocity, and angular acceleration of your leg as a function of time in the form of three separate graphs. Estimate the magnitudes of these quantities.

CHECK YOUR UNDERSTANDING

Angular acceleration is a vector, having both magnitude and direction. How do we denote its magnitude and direction? Illustrate with an example.

Solution

The magnitude of angular acceleration is α and its most common units are rad/s^2. The direction of angular acceleration along a fixed axis is denoted by a + or a – sign, just as the direction of linear acceleration in one dimension is denoted by a + or a – sign. For example, consider a gymnast doing a forward flip. Her angular momentum would be parallel to the mat and to her left. The magnitude of her angular acceleration would be proportional to her angular velocity (spin rate) and her moment of inertia about her spin axis.

Ladybug Revolution

Join the ladybug in an exploration of rotational motion. Rotate the merry-go-round to change its angle, or choose a constant angular velocity or angular acceleration. Explore how circular motion relates to the bug's x,y position, velocity, and acceleration using vectors or graphs.

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Figure 10.7

10.2 Kinematics of Rotational Motion

Just by using our intuition, we can begin to see how rotational quantities like θ, ω, and α are related to one another. For example, if a motorcycle wheel has a large angular acceleration for a fairly long time, it ends up spinning rapidly and rotates through many revolutions. In more technical terms, if the wheel's angular acceleration α is large for a long period of time t, then the final angular velocity ω and angle of rotation θ are large. The wheel's rotational motion is exactly analogous to the fact that the motorcycle's large translational acceleration produces a large final velocity, and the distance traveled will also be large.

Kinematics is the description of motion. The **kinematics of rotational motion** describes the relationships among rotation angle, angular velocity, angular acceleration, and time. Let us start by finding an equation relating ω, α, and t. To determine this equation, we recall a familiar kinematic equation for translational, or straight-line, motion:

$$v = v_0 + at \quad (\text{constant } a) \tag{10.17}$$

Note that in rotational motion $a = a_t$, and we shall use the symbol a for tangential or linear acceleration from now on. As in linear kinematics, we assume a is constant, which means that angular acceleration α is also a constant, because $a = r\alpha$. Now, let us substitute $v = r\omega$ and $a = r\alpha$ into the linear equation above:

$$r\omega = r\omega_0 + r\alpha t. \tag{10.18}$$

The radius r cancels in the equation, yielding

$$\omega = \omega_0 + \alpha t \quad (\text{constant } \alpha), \tag{10.19}$$

where ω_0 is the initial angular velocity. This last equation is a *kinematic relationship* among ω, α, and t —that is, it describes their relationship without reference to forces or masses that may affect rotation. It is also precisely analogous in form to its translational counterpart.

Making Connections

Kinematics for rotational motion is completely analogous to translational kinematics, first presented in One-Dimensional Kinematics. Kinematics is concerned with the description of motion without regard to force or mass. We will find that