Part 3: Electric Potential (Voltage)

University Physics V2 (Openstax): Chapters 7 & 8
Physics for Engineers & Scientists (Giancoli): Chapters 23 & 24

Electrical Potential : $\underbrace{\Delta V}_{\text{Change in electrical potential}} = \underbrace{\frac{\Delta PE}{q}}_{\text{q=test charge}} = \frac{-W}{q} = \underbrace{\frac{-\vec{F}\cdot\vec{d}}{q}}_{\text{Assuming F\&E are constant}} = -\vec{E}\cdot\vec{d}$

- Positive charges accelerate as they move from higher to lower V.
- Negative charges accelerate as they move from lower to higher V.
- Units: Volt (V) $1V = 1\frac{J}{C}$ Note: $1\frac{V}{\mathrm{m}} = 1\frac{J}{mC} = 1\frac{N}{\mathrm{C}}$

Example: Determine the number of particles (each with charge e) that pass between the terminals of a 12.0 V car battery when a 60.0 W headlight burns for an hour.

$$P = \frac{W}{t} \qquad W = Pt = (60.0W)(3600s) = 216kJ$$

$$\Delta V = \frac{W}{q} \qquad q = \frac{W}{\Delta V} = \frac{216kJ}{12.0V} = 18.0kC$$

$$N = \frac{q}{e} = \frac{18000C}{1.60\times10^{-19}C} = 1.13\times10^{23}\,particles$$

Example: Determine the velocity of a proton, which after starting from rest is accelerated through a potential of 252 V.

$$|W| = qV = \frac{1}{2}mv^2$$

$$v = \sqrt{\frac{2qV}{m}} = \sqrt{\frac{2(1.60\times10^{-19}C)(252V)}{1.67\times10^{-27}kg}} = 220km/s$$

The Electron Volt : a new unit of energy

- An electron volt (eV) is the energy acquired by an electron moving through a potential of 1V
- $\mathrm{W} = qV = eV = (1.60\times10^{-19}C)(1\,J/C) = 1.60\times10^{-19}J$
- It's normally used for atomic energy transitions.

Example : The Large Hadron Collider (LHC)

- In the LHC, protons are accelerated to an energy of 7TeV ($7\mathrm{x}10^{12}$ eV).
- This is 7000 times the energy it takes to make a proton from nothing.
- This is equivalent to having the proton pass through a potential of $7\mathrm{x}10^{12}$ V
- The proton beams of the LHC have the kinetic energy of
- 900 cars going 62 mph.

Determining V From E: $dV = -\vec{E}\cdot d\vec{l}$ $\quad \Delta V = -\int_{P_{init}}^{P_{final}} \vec{E}\cdot d\vec{l}$

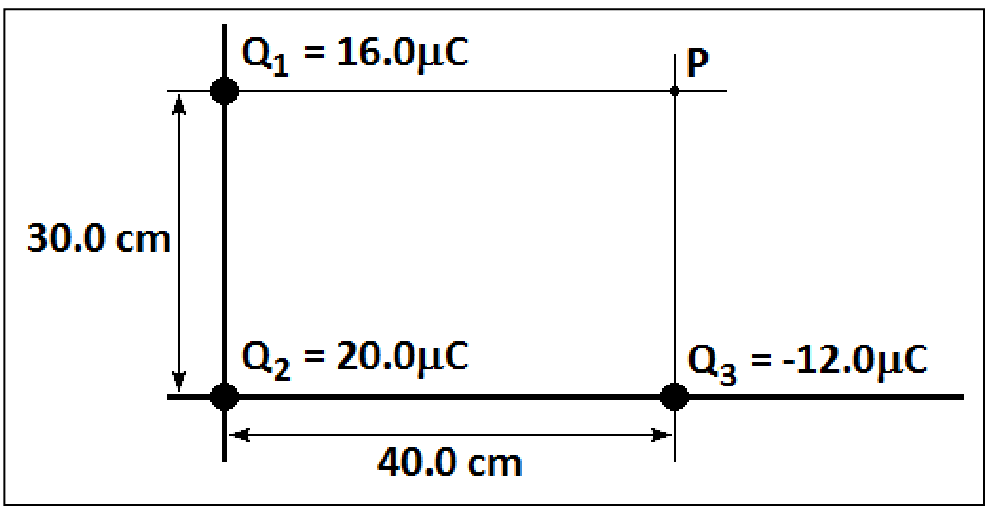
For a Point Charge: $V_{\text{PointCharge}} = k\dfrac{Q}{r}$

- $\Delta V = -\int_{P_{init}}^{P_{final}} \vec{E}\cdot d\vec{l} = -\int_{r_a}^{r_b} \vec{E}\cdot d\vec{r} = -\int_{r_a}^{r_b} k\dfrac{Q}{r^2}dr = k\dfrac{Q}{r_b} - k\dfrac{Q}{r_a}$
- To fix a reference, we set V=0 at r→∞ $\quad V_{\text{PointCharge}} = k\dfrac{Q}{r}$
- V is a scalar (we aren't finding magnitudes) ⇨ Keep the signs on the charges.
- Superposition: Just add up V for multiple charges/sources.

Example: What is the potential at the point (2.25 m, 1.50 m) measured with respect to the origin in a region with a uniform electric field E = (4.00 N/C)î + (2.00 N/C)ĵ ?

$$\Delta V = -\int_{P_{init}}^{P_{final}} \vec{E}\cdot d\vec{l} = -\vec{E}\cdot\left\{\int_{P_{init}}^{P_{final}} d\vec{l}\right\} = -\vec{E}\cdot\vec{l}$$
$$= -\{(4.00\text{ N/C})\hat{\imath} + (2.00\text{ N/C})\hat{\jmath}\}\cdot\{(2.25m)\hat{\imath} + (1.50m)\hat{\jmath}\}$$
$$= -\{(4.00\text{ N/C})(2.25m) + (2.00\text{ N/C})(1.50m)\} = -12.0V$$

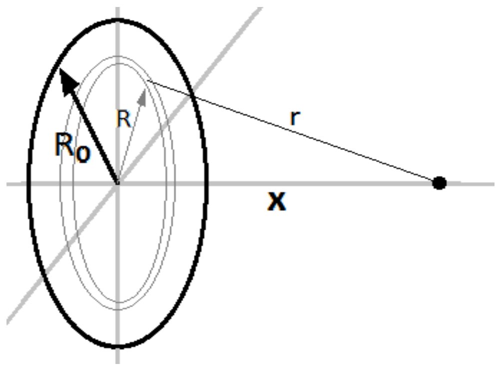
Example: Determine the electric potential at the point P due to the 3 point charges



$$V_{TOT} = V_1 + V_2 + V_3 = k\frac{Q_1}{r_1} + k\frac{Q_2}{r_2} + k\frac{Q_3}{r_3} = k\left(\frac{Q_1}{r_1} + \frac{Q_2}{r_2} + \frac{Q_3}{r_3}\right)$$

$$V_{TOT} = (9.00\times10^9\,Nm^2/C^2)\left\{\frac{16.0\times10^{-6}C}{0.400m} + \frac{20.0\times10^{-6}C}{0.500m} + \frac{-12.0\times10^{-6}C}{0.300m}\right\} = 360kV$$

Example: A thin flat disk of radius R_0 has a uniformly distributed charge Q. Determine the potential at a point P on the axis of the disk, a distance x from the center.



$$dV = k\frac{dQ}{r} \qquad V = \int dV = k\int\frac{dQ}{r}$$

What variable do we integrate over? R (not r)

$$dQ = \sigma dA = \left(\frac{Q}{\pi R_0^2}\right)(2\pi R)dR = \left(\frac{2Q}{R_0^2}\right)RdR$$

$$dV = k\frac{dQ}{r} = \left(\frac{2kQ}{R_0^2}\right)\frac{RdR}{r} = \left(\frac{2kQ}{R_0^2}\right)\frac{RdR}{\sqrt{R^2+x^2}}$$

Can we just integrate dV? Yes! dV is a scalar!

$$V = \int dV = \left(\frac{2kQ}{R_0^2}\right)\int_0^{R_0}\frac{RdR}{\sqrt{R^2+x^2}}$$

$$U = [R^2 + x^2]^{1/2} \qquad dU = \frac{1}{2}[R^2 + x^2]^{-1/2}\,2RdR$$

$$V = \left(\frac{2kQ}{R_0^2}\right)\int_0^{R_0} dU = \left(\frac{2kQ}{R_0^2}\right)[U]_{R=0}^{R=R_0} = \left(\frac{2kQ}{R_0^2}\right)\left\{\sqrt{R^2+x^2}\right\}_{R=0}^{R=R_0} = \left(\frac{2kQ}{R_0^2}\right)\left\{\sqrt{R_0^2+x^2} - x\right\}$$

E from V

- $dV = -\vec{E}\cdot d\vec{l} \quad E_l = -\dfrac{\partial V}{\partial l}$
- $E_x = -\dfrac{\partial V}{\partial x}, E_y = -\dfrac{\partial V}{\partial y}, E_z = -\dfrac{\partial V}{\partial z}$

Example: In a certain region of space the electric potential is given by V(x,y,z) = y^2 + 2.5xy – 3.5xyz. Determine the electric field.

$$E_x = -\frac{\partial V}{\partial x} = -(2.5y - 3.5yz) = 3.5yz - 2.5y$$

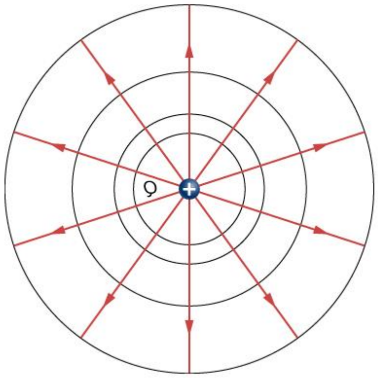
$$E_y = -\frac{\partial V}{\partial y} = -(2y + 2.5x - 3.5xz) = 3.5xz - 2.5x - 2y$$

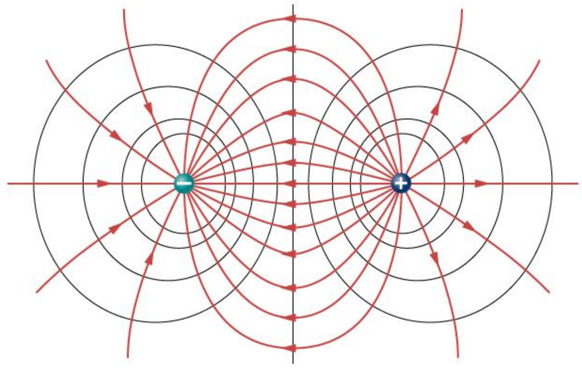
$$E_z = -\frac{\partial V}{\partial z} = -(-3.5xy) = 3.5xy$$

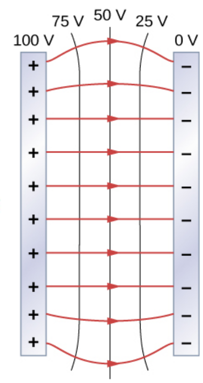
$$\vec{E} = (3.5yz - 2.5y)\hat{i} + (3.5xz - 2.5x - 2y)\hat{j} + (3.5xy)\hat{k}$$

<u>Equipotential Surfaces</u> (Every point in the surface is at the same potential)

- No work is done by electrical forces when moving along an equipotential surface.
- The electric field is always $\perp$ to the equipotential surface.





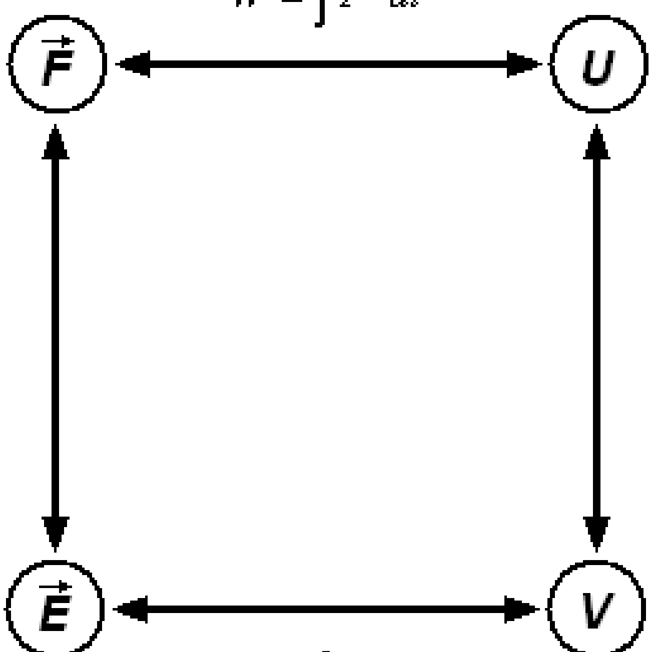


<u>Roadmap</u>

$$\left|\vec{F}\right| = k_c \frac{Q_1 Q_2}{r^2}$$

$$W = \int \vec{F} \cdot d\vec{l}$$

$$U = k_c \frac{Q_1 Q_2}{r}$$



$$\vec{E} = \frac{\vec{F}}{q}$$

$$\vec{F} = q\vec{E}$$

$$V_a = \frac{PE_a}{q} = -\frac{W}{q}$$

$$\Delta PE = q \cdot \Delta V$$

$$\left|\vec{E}\right| = k_c \frac{Q}{r^2}$$

$$V = -\int \vec{E} \cdot d\vec{l}$$

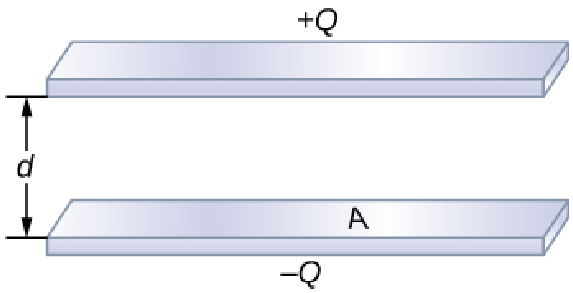
$$V = k_c \frac{Q}{r}$$

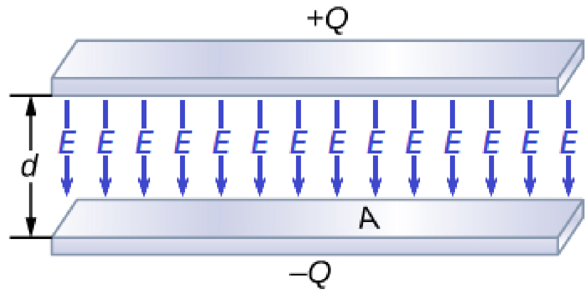
$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z}$$

Capacitance: $Q = CV$

- Q = Charge
- C = Capacitance
- V = Electric Potential

- Units: Farad 1 F = 1 C/V





Parallel Plate Capacitor

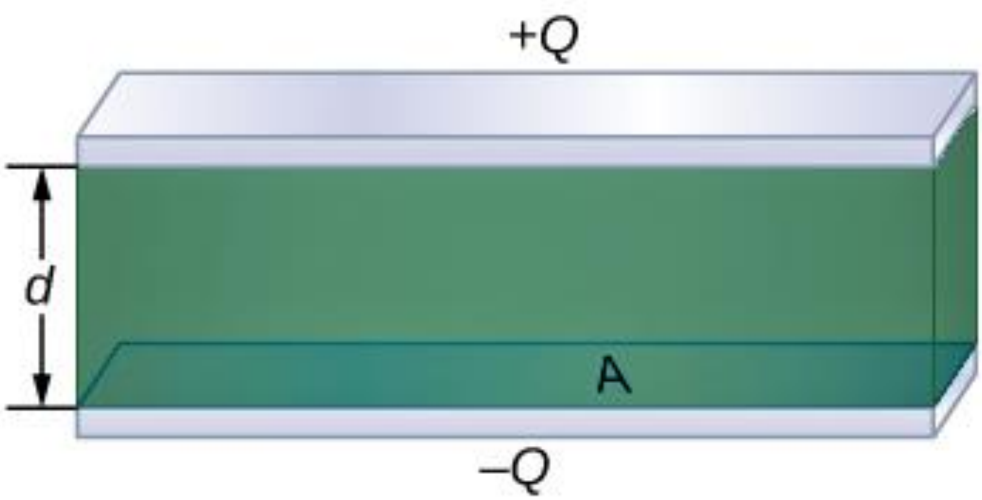
Electric field at surface of conductor: $E = \dfrac{\sigma}{\varepsilon_0} = \dfrac{Q}{\varepsilon_0 A}$

E is constant.

$$V = -\vec{E} \cdot \vec{d} = Ed = \frac{Qd}{\varepsilon_0 A} = \left(\frac{d}{\varepsilon_0 A}\right) Q \quad Q = \left(\frac{\varepsilon_0 A}{d}\right) V = CV \quad \mathrm{C} = \frac{\varepsilon_0 A}{d}$$

Dielectrics: A **dielectric** is an insulating material placed between the capacitor plates.

Let C_0 be the capacitance without the dielectric, and C the capacitance with the dielectric.



k = "dielectric constant" $\qquad \varepsilon > \varepsilon_0$

$\varepsilon_0 \to \varepsilon \quad \varepsilon = \kappa \varepsilon_0 \qquad \kappa > 1$

$\mathrm{C} = \frac{\varepsilon A}{d} = \frac{\kappa \varepsilon_0 A}{d} = \kappa C_0 \qquad \mathrm{C} > C_0$

Example: What voltage is required to store $7.20\text{x}10^{-5}$C of charge on the plates of a 6.00μF capacitor?

$$Q = CV \quad V = \frac{Q}{C} = \frac{7.20\times10^{-5}\,C}{6.00\times10^{-6}\,F} = 12.0\,\text{V}$$

Example: A parallel plate capacitor has a capacitance of 7.00μF when filled with a dielectric. The area of each plate is 1.50m^2 and the separation between the plates is $1.00\text{x}10^{-5}$m. What is the dielectric constant of the dielectric material?

$$C = \frac{\kappa\varepsilon_0 A}{d} \quad \kappa = \frac{dC}{\varepsilon_0 A} = \frac{(1.00\times10^{-5}\,m)(7.00\times10^{-6}\,F)}{\left[8.85\times10^{-12}C^2/(Nm^2)\right](1.50m^2)} = 5.3$$

Energy Storage in Capacitors: $PE = \frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{Q^2}{2C}$

- $W = \int_0^Q V dq = \frac{1}{C}\int_0^Q q dq = \frac{Q^2}{2C}$
- $\frac{Q^2}{2C} = \frac{1}{2}Q\left(\frac{Q}{C}\right) = \frac{1}{2}QV \qquad \frac{1}{2}QV = \frac{1}{2}(CV)V = \frac{1}{2}CV^2$

Energy Density of Electric Field

- Look at a parallel plate capacitor
- $PE = \frac{1}{2}CV^2 = \frac{1}{2}\left(\frac{\varepsilon A}{d}\right)(Ed)^2 = \frac{1}{2}\varepsilon AdE^2 = \frac{1}{2}\varepsilon(Ad)E^2$
- $Energy\ Density = \frac{Energy}{Volume} = \frac{1}{2}\varepsilon E^2$

Example: A 6.00 μF capacitor is charged to 12.0 V. (a) How much charge and energy are stored in the capacitor. (b) If the plates of the charged capacitor are then connected by a pair of thin conductors to the plates of a second, uncharged 12.0 μF capacitor, what is the shared electrical potential once equilibrium has been achieved and how much energy is stored by the pair?

$$Q = CV = (6.00\mu F)(12.0V) = 72.0\mu C$$

$$PE = \frac{1}{2}CV^2 = \frac{1}{2}(6.00\mu F)(12.0V)^2 = 432\mu J$$

$$V_1 = V_2 = V \quad Q_0 = Q_1 + Q_2$$

$$Q_0 = Q_1 + Q_2 = C_1V + C_2V = (C_1 + C_2)V$$

$$V = \frac{Q_0}{C_1 + C_2} = \frac{72.0\mu C}{(6.00\mu F + 12.0\mu F)} = 4.00V$$

$$PE = \frac{1}{2}C_1V^2 + \frac{1}{2}C_2V^2 = \frac{1}{2}(C_1 + C_2)V^2 = \frac{1}{2}(6.00\mu F + 12.0\mu F)(4.00V)^2 = 144\mu J$$

Example: A 6.00 μF capacitor is held at a constant electrical potential of 12.0 V. A dielectric (k = 3.00) is then inserted between the capacitors plates. Assuming that friction is negligible, how much work must be done to insert the dielectric?

$$W_{done} = PE_{final} - PE_{init} = \frac{1}{2}C_F V^2 - \frac{1}{2}CV^2 = \frac{1}{2}(C_F - C)V^2$$

$$= \frac{1}{2}(\kappa C - C)V^2 = \frac{1}{2}(\kappa - 1)CV^2$$

$$= \frac{1}{2}(3.00 - 1)(6.00\mu F)(12.0V)^2 = 864\mu J$$

Exercises

1. An electron moves across a 5.00 V electric potential. Determine the magnitude of the electron's change in kinetic energy.

2. The plates of a parallel plate capacitor are separated by 1.00 mm. Determine the magnitude of the electric field between the plates when the voltage across the capacitor is 5.00 V.

3. A proton is headed directly towards an object with a charge that is 8350 times as large as the proton. When the proton is far away from the object its velocity is 1200 m/s. How close does it get to the object?

4. Two 25.0 μC charges are fixed on the x-axis. One is placed at the origin, and the other is placed at x = 6.00 m. Determine the electric potential at x = 3.00 m on the x-axis.

5. Two charges are placed on an xy-plane. A 5.00 μC charge (Q1) is fixed at the origin. A second charge (Q2 = -10.0 μC) is fixed at the point (7.00 m, 1.00 m). Determine the electric potential at the point (3.00 m, 4.00 m).

6. The plates in an air-filled parallel plate capacitor are squares with sides of length 1.00 cm separated by 88.5 nm. It is charged so that the voltage across it is 4.425 V. Determine (A) The capacitance of the capacitor, (B) The charge stored on the capacitor, (C) The magnitude of the electric field between the plates.

7. The capacitor from the previous problem is now filled with a dielectric (κ = 4.00). Determine (A) The capacitance of the capacitor, (B) The charge stored on the capacitor, (C) The magnitude of the electric field between the plates.

8. What is the potential at the point (3.00 m, 2.00 m) measured with respect to the origin in a region with a non-uniform electric field $\vec{E} = \alpha x\hat{\imath} + \beta y^2\hat{\jmath}$, where α = – 4.00 N/(C·m) and β = 3.00 N/(C·m^2)?

9. A 3.00 m long wire that holds a net charge of 5.00 μC is laid along the y-axis from y_1 = –1.00 m to y_2 = 2.00 m. Set up the integral to determine the electric potential at x_0 = 5.00 m on the x-axis.

Exercise Solutions

1. An electron moves across a 5.00 V electric potential. Determine the magnitude of the electron's change in kinetic energy.

$$|\Delta KE| = |W| = qV = (1.60 \times 10^{-19}\,C)(5.00\,V) = 8.00 \times 10^{-19}\,J$$

2. The plates of a parallel plate capacitor are separated by 1.00 mm. Determine the magnitude of the electric field between the plates when the voltage across the capacitor is 5.00 V.

Inside a parallel plate capacitor, E is constant. $|E| = \frac{V}{d} = \frac{5.00\,V}{1.00 \times 10^{-3}\,m} = 5.00\frac{kV}{m}$

3. A proton is headed directly towards an object with a charge that is 8350 times as large as the proton. When the proton is far away from the object its velocity is 1200 m/s. How close does it get to the object?

When far away from a charge, the potential energy approaches zero. So all the energy is kinetic. At it's closest point to the charge, the proton will come to rest before being pushed away. When it comes to rest, it will only have electric potential energy.

$$KE = \frac{1}{2}mv^2 = qV = eV = e\left(\frac{kQ}{r}\right) = e\left(\frac{8350ke}{r}\right) = \frac{8350ke^2}{r} = \frac{1}{2}mv^2$$

$$r = \frac{16700ke^2}{mv^2} = \frac{16700\left(9.00 \times 10^9\,\frac{N\cdot m^2}{C^2}\right)(1.60 \times 10^{-19}\,C)^2}{(1.67 \times 10^{-27}\,C)\left(1200\,\frac{m}{s}\right)^2} = 16.0\,m$$

4. Two 25.0 μC charges are fixed on the x-axis. One is placed at the origin, and the other is placed at x = 6.00 m. Determine the electric potential at x = 3.00 m on the x-axis.

$$V = V_1 + V_2 = \frac{kQ_1}{r_1} + \frac{kQ_2}{r_2} = \frac{2kQ}{r} = \frac{2\left(9.00 \times 10^9\,\frac{N\cdot m^2}{C^2}\right)(25.0 \times 10^{-6}\,C)}{3.00\,m} = 150\,kV$$

Note that $Q = Q_1 = Q_2 = 25.0\,\mu C$ and $r = r_1 = r_2 = 3.00\,m$

5. Two charges are placed on an xy-plane. A 5.00 μC charge (Q1) is fixed at the origin. A second charge (Q2 = -10.0 μC) is fixed at the point (7.00 m, 1.00 m). Determine the electric potential at the point (3.00 m, 4.00 m).

$$r_1 = \sqrt{(3.00\,m)^2 + (4.00\,m)^2} = 5.00\,m \qquad r_2 = \sqrt{(7.00m - 3.00\,m)^2 + (1.00\,m - 4.00\,m)^2} = 5.00\,m$$

$$V = V_1 + V_2 = \frac{kQ_1}{r_1} + \frac{kQ_2}{r_2} = \frac{k}{r}(Q_1 + Q_2) = \frac{\left(9.00 \times 10^9\,\frac{N\cdot m^2}{C^2}\right)}{5.00\,m}(5.00 \times 10^{-6}\,\mathrm{C} - 10.0 \times 10^{-6}\,\mathrm{C}) = -9.00\,kV$$

6. The plates in an air-filled parallel plate capacitor are squares with sides of length 1.00 cm separated by 88.5 nm. It is charged so that the voltage across it is 4.425 V. Determine (A) The capacitance of the capacitor, (B) The charge stored on the capacitor, (C) The magnitude of the electric field between the plates.

   A. $C = \frac{\epsilon_0 A}{d} = \frac{\left(8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2}\right)(0.0100\, m)^2}{88.5 \times 10^{-9}\, m} = 10.0\, nF$

   B. $Q = CV = (10.0\, nF)(4.425\text{ V}) = 44.25\, nC$

   C. *Inside a parallel plate capacitor, E is constant.* $|E| = \frac{V}{d} = \frac{4.425\, V}{88.5 \times 10^{-9}\, m} = 50.0 \frac{MV}{m}$

7. The capacitor from the previous problem is now filled with a dielectric ($\kappa = 4.00$). Determine (A) The capacitance of the capacitor, (B) The charge stored on the capacitor, (C) The magnitude of the electric field between the plates.

   A. $C = \kappa C_0 = (4.00)(10.0\, nF) = 40.0\, nF$

   B. $Q = CV = (40.0\, nF)(4.425\text{ V}) = 177\, nC$

   C. $|E| = \frac{V}{d} = \frac{4.425\, V}{88.5 \times 10^{-9}\, m} = 50.0 \frac{MV}{m}$

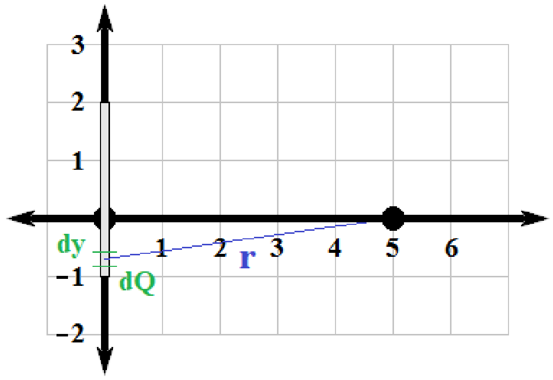
8. What is the potential at the point (3.00 m, 2.00 m) measured with respect to the origin in a region with a non-uniform electric field $\vec{E} = \alpha x \hat{\imath} + \beta y^2 \hat{\jmath}$, where $\alpha = -4.00$ N/(C·m) and $\beta = 3.00$ N/(C·m^2)?

$$V = -\int \vec{E} \cdot d\vec{l} = -\int (\alpha x \hat{\imath} + \beta y^2 \hat{\jmath}) \cdot (dx \hat{\imath} + dy \hat{\jmath}) = -\int_0^{3.00\, m} \alpha x dx - \int_0^{2.00\, m} \beta y^2 dy$$

$$V = -\left[\frac{1}{2}\alpha x^2\right]_{x=0}^{x=3.00\, m} - \left[\frac{1}{3}\beta y^3\right]_{y=0}^{y=2.00\, m} = -\frac{1}{2}\left(-4.00 \frac{N}{C \cdot m}\right)(3.00\, m)^2 - \frac{1}{3}\left(3.00 \frac{N}{C \cdot m^2}\right)(2.00\, m)^2$$

$$V = 18.0\, V - 4.00\, V = 14.0\, V$$

9. A 3.00 m long wire that holds a net charge of 5.00 μC is laid along the y-axis from $y_1 = -1.00$ m to $y_2 = 2.00$ m. Set up the integral to determine the electric potential at $x_0 = 5.00$ m on the x-axis.



$$dQ = \lambda dy = \frac{Q}{L} dy$$

$$dV = k\frac{dQ}{r} = \frac{kQdy}{rL} = \frac{kQ}{L}\frac{dy}{\sqrt{y^2 + {x_0}^2}}$$

$$V = \int dV = \frac{kQ}{L}\int_{-1.00\,m}^{2.00\,m}\frac{dy}{\sqrt{y^2 + {x_0}^2}}$$

With Q = 5.00 μC, L = 3.00 m, and x_0 = 5.00 m