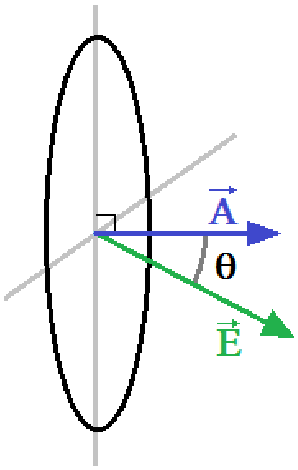
Part 2: Gauss's Law

University Physics V2 (Openstax): Chapter 6
Physics for Engineers & Scientists (Giancoli): Chapter 22

Electric Flux:
$$\underbrace{\Phi_E}_{\substack{Electric\\Flux}} = \vec{E} \cdot \underbrace{\vec{A}}_{Area}$$



Analogy: Rain accumulating in a barrel.

If 2 inches of rain falls vertically and the opening at the top of a barrel is 100 in^2, how much rain falls in the barrel?

(2 inches)(100 in^2) = 200 in^3

What if the rain's path makes a 30° angle with the vertical?

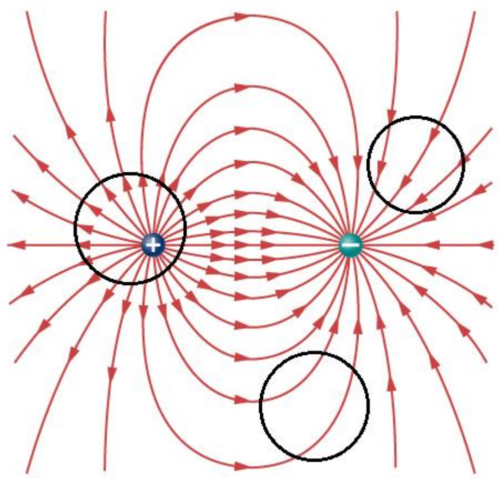
(2 inches)(100 in^2)Cos (30°) = 173 in^3

This is a dot product.

"Flux" is very similar in meaning to "flow". What's flowing in F_E? Nothing. That's why it's "flux" and not "flow".

Gauss's Law:
$$\underbrace{\Phi_E}_{\substack{\text{Electric}\\\text{Flux}}} = \underbrace{\oint \vec{E} \cdot \vec{A}}_{\substack{\text{Summed Over}\\\text{Closed Surface}}} = \underbrace{\frac{Q_{ENC}}{\varepsilon_0}}_{\substack{\text{Enclosed Charge}\\\text{(divided by a const)}}}$$

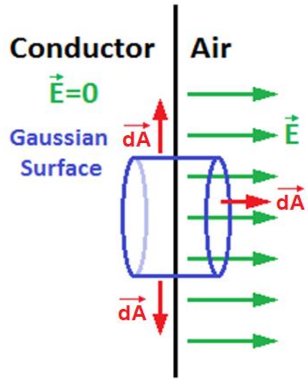
- "Permittivity of Free Space" $\varepsilon_0 = 8.85\text{x}10^{-12} C^2/(Nm^2)$ $\varepsilon_0 = \dfrac{1}{4\pi k}$



Example: The geometric center of a cube is located at the origin. If a 30.0 nC point charge is also at the origin, what is the electric flux that passes through one of the cube's faces?

$$\Phi_E = \frac{1}{6} \cdot \frac{Q_{ENC}}{\varepsilon_0} = \frac{30.0\times10^{-9}C}{(6)[8.85\times10^{-12}C^2/(Nm^2)]} = 565\frac{Nm^2}{C}$$

Example: A conductor has a surface charge density of σ = 3.25μC/m². Determine the electric field at the surface.



$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{ENC}}{\varepsilon_0}$$

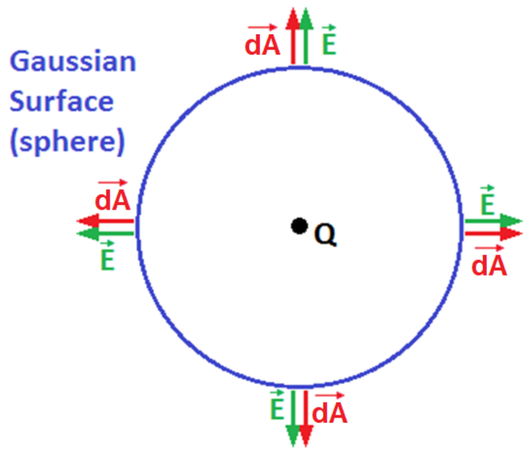
Back Side (circle): $\vec{E}=0 \quad \vec{E}\cdot\vec{A}=0$

Sides (tube): $\vec{A}\perp\vec{E} \quad \vec{A}\cdot\vec{E}=0$

Front Side (circle): $\vec{A}\parallel\vec{E} \quad \vec{E}\cdot\vec{A}=EA$

$$EA=\frac{Q_{ENC}}{\varepsilon_0} \qquad E=\frac{Q_{ENC}}{\varepsilon_0 A}=\frac{\sigma}{\varepsilon_0}=\frac{3.25\times10^{-6}C/m^2}{8.85\times10^{-12}C^2/(N\cdot m^2)}=367kN/C$$

Example: Find the Electric Field due to a point charge Q using Gauss's Law.



$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{ENC}}{\varepsilon_0}$$

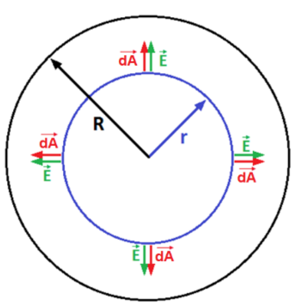
$$d\vec{A}||\vec{E} \quad \vec{E}\cdot d\vec{A} = E\cdot dA$$

$$\oint E\cdot dA = E\oint dA = E(4\pi r^2)$$

$$E(4\pi r^2)=\frac{Q}{\varepsilon_0}$$

$$E=\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}=k\frac{Q}{r^2}$$

Example: A non-conducting sphere of radius R has a non-uniform charge distribution, ρ = α + βr, where r is the distance from the sphere's center. Determine the electric field as a function of r.



$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{ENC}}{\varepsilon_0}$$

We have spherical symmetry.

$$d\vec{A}||\vec{E} \quad \vec{E}\cdot d\vec{A} = E\cdot dA$$

$$\int E\cdot dA = E\int dA = E(4\pi r^2)$$

For r < R, $Q_{ENC} = \int \rho dV = \int_0^r (\alpha+\beta r)(4\pi r^2)(dr)$

$$Q_{ENC} = 4\pi\int_0^r (\alpha r^2+\beta r^3)dr = 4\pi\left[\frac{1}{3}\alpha r^3+\frac{1}{4}\beta r^4\right]_{r=0}^{r=r} = \frac{4\pi}{3}\alpha r^3+\pi\beta r^4$$

$$E = \frac{Q_{ENC}}{4\pi\varepsilon_0 r^2} \qquad E = \frac{4\pi\alpha r^3/3+\pi\beta r^4}{4\pi\varepsilon_0 r^2} \qquad E = \frac{\alpha r}{3\varepsilon_0} + \frac{\beta r^2}{4\varepsilon_0}$$

For r > R, $Q_{ENC} = \int \rho dV = \int_0^R (\alpha + \beta r)(4\pi r^2)(dr)$

The only difference between this and our previous integration is that now we stop integrating at R, which is the outer limit of where our charge is found.

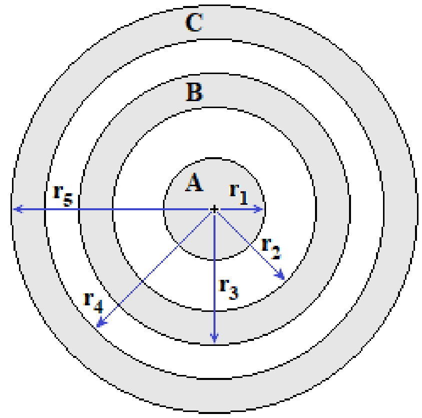
$$Q_{ENC} = 4\pi \int_0^R (\alpha r^2 + \beta r^3) dr = 4\pi \left[\frac{1}{3}\alpha r^3 + \frac{1}{4}\beta r^4\right]_{r=0}^{r=R} = \frac{4\pi}{3}\alpha R^3 + \pi\beta R^4$$

$$E = \frac{Q_{ENC}}{4\pi\varepsilon_0 r^2} \qquad E = \frac{4\pi\alpha R^3/3+\pi\beta R^4}{4\pi\varepsilon_0 r^2} \qquad E = \frac{\alpha R^3}{3\varepsilon_0 r^2} + \frac{\beta R^4}{4\varepsilon_0 r^2}$$

$$E = \begin{cases} \dfrac{\alpha r}{3\varepsilon_0} + \dfrac{\beta r^2}{4\varepsilon_0}, \text{for r} < R \\ \dfrac{\alpha R^3}{3\varepsilon_0 r^2} + \dfrac{\beta R^4}{4\varepsilon_0 r^2}, \text{for r} \geq R \end{cases}$$

Exercises

1. A conducting sphere (A) has a radius $r_1 = 10.0$ cm and a net charge of 22 nC. The sphere (A) is surrounded by two conducting spherical shells. The first shell (B) has an inner radius of $r_2 = 20.0$ cm, an outer radius of $r_3 = 25.0$ cm, and a charge of 174 nC. The second shell (C) has an inner radius of $r_4 = 30.0$ cm, an outer radius of $r_5 = 35.0$ cm, and a charge of – 196 nC. All of these objects are in static equilibrium.



   A. Determine the charge distribution on the surface of the conducting sphere (A).

   B. Determine the net charge on the inner surface of the first spherical shell (B).

   C. Determine the net charge on the outer surface of the first spherical shell (B).

   D. Determine the charge distribution on the outer surface of the second spherical shell (C).

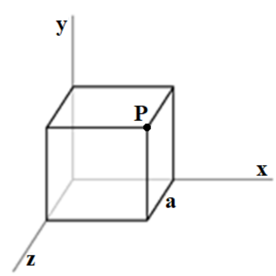
   E. Determine the magnitude of the electric field at r = 28.0 cm.

   F. Determine the magnitude of the electric field at r = 22.0 cm.

2. A non-conducting sphere of radius 20.0 cm has 80.0 nC of charge uniformly distributed inside it. Determine the magnitude of the electric field a distance of 10.0 cm from the center of the sphere.

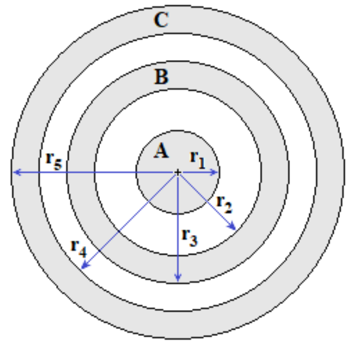
3. A very long cylinder is 10.0 cm in radius and has a non-uniform charge density $\rho = \alpha + \beta r^2$, where $\alpha = 88.5$ nC/m^3 and $\beta = 4425$ nC/m^5. Determine the magnitude of the electric field at a distance of 20.0 cm from the center of the shell.

4. A cube with sides of length a = 2.00 m is located in a coordinate system with one vertex at the origin and the opposite vertex at the point P (2.00 m, 2.00 m, 2.00 m). In this region of space there is an electric field given by $\vec{E} = (\alpha x)\hat{\imath} + (\beta)\hat{\jmath} + (\gamma z^2)\hat{k}$, where $\alpha = 7.00\,\frac{N}{C\cdot m}$, $\beta = 5.00\,\frac{N}{C}$, and $\gamma = 3.00\,\frac{N}{C\cdot m^2}$. Determine the net charge inside the cube.



Exercise Solutions

1. A conducting sphere (A) has a radius r_1 = 10.0 cm and a net charge of 22 nC. The sphere (A) is surrounded by two conducting spherical shells. The first shell (B) has an inner radius of r_2 = 20.0 cm, an outer radius of r_3 = 25.0 cm, and a charge of 174 nC. The second shell (C) has an inner radius of r_4 = 30.0 cm, an outer radius of r_5 = 35.0 cm, and a charge of – 196 nC. All of these objects are in static equilibrium.



A. Determine the charge distribution on the surface of the conducting sphere (A).

As no net charge can stay inside a conductor in static equilibrium, it must be on the surface.

$$\sigma = \frac{Q}{A} = \frac{Q}{4\pi r_1^2} = \frac{(22\ nC)}{4\pi(0.100\ m)^2} = 175\frac{nC}{m^2}$$

B. Determine the net charge on the inner surface of the first spherical shell (B).

-22 nC.

The electric field inside the shell is zero. If we draw a Gaussian sphere just slightly bigger than r_2 then the flux through it will be zero, as E=0 on that surface (inside the conductor). If the flux is zero, the net enclosed charge must also be zero. As there are 22 nC on the inner sphere, there must be -22 nC on the inner surface of conducting shell B.

C. Determine the net charge on the outer surface of the first spherical shell (B).

The total charge on shell B (174 nC) is distributed over the inner and outer surfaces. So the sum of the charges on those two surfaces must be the total charge.

$Q_B = Q_{B-In} + Q_{B-Out}$ $\quad$ $Q_{B-Out} = Q_B - Q_{B-In} = 174\ nC - (-22\ nC) = 196\ nC$

D. Determine the charge distribution on the outer surface of the second spherical shell (C).

0

By the same logic as part B, the inner surface of shell C must have the opposite charge as the outer surface of shell B. As the outer surface of shell B holds 196 nC, the inner surface of shell C must have -196 nC. The total charge on shell C (-196 nC) is distributed over the inner and outer surfaces. So the sum of the charges on those two surfaces must be the total charge. As the total charge is found on the inner surface, there is none left for the outer surface.

E. Determine the magnitude of the electric field at r = 28.0 cm.

This radius falls in the gap between shells B and C. Using Gauss's Law on a spherically symmetric object always leads to an electric field that is identical to that the would be created by placing all of the enclosed charge as a point charge at the center.

$$E = k\frac{Q_{Enc}}{r^2} = \left(9.00 \times 10^9 \ \frac{N{\cdot}m^2}{C^2}\right)\frac{(196 \times 10^{-9}\ C)}{(0.280\ m)^2} = 22.5\frac{kN}{C}$$

F. Determine the magnitude of the electric field at r = 22.0 cm.

0

This is inside shell B, a conductor in static equilibrium. The electric field is always zero inside a conductor in static equilibrium.

2. A non-conducting sphere of radius 20.0 cm has 80.0 nC of charge uniformly distributed inside it. Determine the magnitude of the electric field a distance of 10.0 cm from the center of the sphere.

Gauss's Law on a spherically symmetric object always leads to an electric field that is identical to that the would be created by placing all of the enclosed charge as a point charge at the center. As the charge is uniformly distributed, we can find the enclosed charge using the charge density.

$$Q_{Enc} = \rho V_{Enc} = \left(\frac{Q}{V_{Tot}}\right)(V_{Enc}) = \left(\frac{Q}{\frac{4}{3}\pi R^3}\right)\left(\frac{4}{3}\pi r^3\right) = Q\left(\frac{r}{R}\right)^3 = (80.0\ nC)\left(\frac{10.0\ cm}{20.0\ cm}\right)^3 = 10.0\ nC$$

$$E = k\frac{Q_{Enc}}{r^2} = \left(9.00 \times 10^9 \ \frac{N{\cdot}m^2}{C^2}\right)\frac{(10 \times 10^{-9}\ C)}{(0.100\ m)^2} = 9.00\frac{kN}{C}$$

3. A very long cylinder is 10.0 cm in radius and has a non-uniform charge density $\rho = \alpha + \beta r^2$, where α = 88.5 nC/m^3 and β = 4425 nC/m^5. Determine the magnitude of the electric field at a distance of 20.0 cm from the center of the shell.

Make a cylinder of radius R_2 = 20.0 cm and length L as your gaussian surface. Then apply Gauss's Law. By symmetry, the electric field should point radially outward. Consequently, there will be no flux through the circular ends of the cylinder, and $\vec{E}$ on the surface will always be parallel to $d\vec{A}$.

$$\oint \vec{E} \cdot d\vec{A} = \oint E dA = E\oint dA = E \cdot (2\pi R_2 L)$$

To get the enclosed charge, we will have to integrate $dQ = \rho dV$. As ρ has a dependence on r, our dV can only include parts of the cylinder that have the same value of r. Consequently we will sum up the charge in small cylindrical shells of radius r and thickness dr over the radius (R_1) of the cylinder.

$$Q_{Enc} = \int_0^{R_1} \rho dV = \int_0^{R_1} \rho \cdot (2\pi r \cdot L \cdot dr) = \int_0^{R_1} (\alpha + \beta r^2) \cdot (2\pi r \cdot L \cdot dr)$$

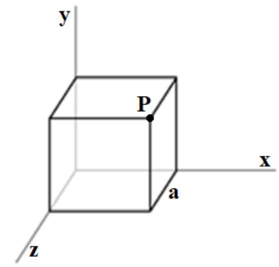
$$Q_{Enc} = 2\pi L \int_0^{R_1} (\alpha r + \beta r^3) dr = 2\pi L\left[\frac{1}{2}\alpha r^2 + \frac{1}{4}\beta r^4\right]_{r=0}^{r=R_1} = 2\pi L\left(\frac{1}{2}\alpha {R_1}^2 + \frac{1}{4}\beta {R_1}^4\right)$$

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{Enc}}{\epsilon_0} \qquad E \cdot (\cancel{2\pi} R_2 \cancel{L}) = \frac{\cancel{2\pi L}}{\epsilon_0}\left(\frac{1}{2}\alpha {R_1}^2 + \frac{1}{4}\beta {R_1}^4\right)$$

$$E = \frac{2\alpha {R_1}^2 + \beta {R_1}^4}{4R_2\epsilon_0} = \frac{2\left(88.5\times10^{-9}\frac{C}{m^3}\right)(0.100\ m)^2 + \left(4425\times10^{-9}\frac{C}{m^5}\right)(0.100\ m)^4}{4(0.200\ m)\left(8.85\ \times10^{-12}\ \frac{C^2}{N\cdot m^2}\right)}$$

$$E = 312.5\frac{N}{C}$$

4. A cube with sides of length a = 2.00 m is located in a coordinate system with one vertex at the origin and the opposite vertex at the point P (2.00 m, 2.00 m, 2.00 m). In this region of space there is an electric field given by $\vec{E} = (\alpha x)\hat{\imath} + (\beta)\hat{\jmath} + (\gamma z^2)\hat{k}$, where $\alpha = 7.00\frac{N}{C\cdot m}$, $\beta = 5.00\frac{N}{C}$, and $\gamma = 3.00\frac{N}{C\cdot m^2}$. Determine the net charge inside the cube.



$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{Enc}}{\epsilon_0}$ $\qquad Q_{Enc} = \epsilon_0 \oint \vec{E}\cdot d\vec{A}$

Using Gauss's Law, to integrate over the surface, we must integrate over each of the 6 faces of the cube and then sum the flux from each.

Right Face: $\quad d\vec{A} = dA\hat{\imath} \quad \Phi_1 = \int \vec{E}\cdot d\vec{A} = \int\left[(\alpha x)\hat{\imath} + (\beta)\hat{\jmath} + (\gamma z^2)\hat{k}\right]\cdot dA\hat{\imath} = \int \alpha x\cdot dA$

As x is constant over the surface (x=a), it can be pulled out of the integral. Integrating dA just gives the area of the face (a^2).

$$\Phi_1 = \alpha a\int dA = \alpha a A = \alpha a^3$$

Left Face: $\quad d\vec{A} = -dA\hat{\imath} \quad \Phi_2 = \int \vec{E}\cdot d\vec{A} = \int\left[(\alpha x)\hat{\imath} + (\beta)\hat{\jmath} + (\gamma z^2)\hat{k}\right]\cdot -dA\hat{\imath} = \int -\alpha x\cdot dA$

$\Phi_2 = 0.$ *As x=0 on the surface, the electric flux (Φ_2) is zero.*

Front Face: $\quad d\vec{A} = dA\hat{k} \quad \Phi_3 = \int \vec{E}\cdot d\vec{A} = \int\left[(\alpha x)\hat{\imath} + (\beta)\hat{\jmath} + (\gamma z^2)\hat{k}\right]\cdot dA\hat{k} = \int \gamma z^2\cdot dA$

As z is constant over the surface (z=a), it can be pulled out of the integral. Integrating dA just gives the area of the face (a^2).

$$\Phi_3 = \gamma a^2\int dA = \gamma a^2 A = \gamma a^4$$

Back Face: $\quad d\vec{A} = -dA\hat{k} \quad \Phi_4 = \int \vec{E}\cdot d\vec{A} = \int\left[(\alpha x)\hat{\imath} + (\beta)\hat{\jmath} + (\gamma z^2)\hat{k}\right]\cdot -dA\hat{k} = \int -\gamma z^2\cdot dA$

$\Phi_4 = 0.$ *As z=0 on the surface, the electric flux (Φ_2) is zero.*

Top Face: $\quad d\vec{A} = dA\hat{\jmath} \quad \Phi_5 = \int \vec{E}\cdot d\vec{A} = \int\left[(\alpha x)\hat{\imath} + (\beta)\hat{\jmath} + (\gamma z^2)\hat{k}\right]\cdot dA\hat{\jmath} = \int \beta\cdot dA$

$\Phi_5 = \beta\int dA = \beta a^2$

Bottom Face: $\quad d\vec{A} = -dA\hat{\jmath} \quad \Phi_6 = \int \vec{E}\cdot d\vec{A} = \int\left[(\alpha x)\hat{\imath} + (\beta)\hat{\jmath} + (\gamma z^2)\hat{k}\right]\cdot -dA\hat{\jmath} = \int -\beta\cdot dA$

$\Phi_6 = -\beta\int dA = -\beta a^2.$

$$Q_{Enc} = \epsilon_0 \oint \vec{E} \cdot d\vec{A} = \epsilon_0(\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5 + \Phi_6)$$

$$Q_{Enc} = \epsilon_0(\alpha a^3 + 0 + \gamma a^4 + 0 + \beta a^2 - \beta a^2) = \epsilon_0(\alpha a^3 + \gamma a^4)$$

$$Q_{Enc} = \left(8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2}\right)\left[\left(7.00 \frac{N}{C \cdot m}\right)(2.00\ m)^3 + \left(3.00 \frac{N}{C \cdot m^2}\right)(2.00\ m)^4\right]$$

$$Q_{Enc} = 920.4\ pC$$