

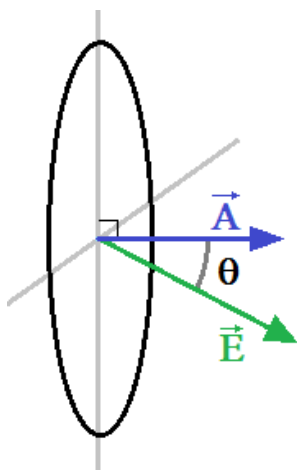
## Part 2: Gauss's Law

College Physics (Openstax): ???

Physics: Principles with Applications (Giancoli): Chapter 16

### Electric Flux:

$$\underbrace{\Phi_E}_{\text{Electric Flux}} = \underbrace{\vec{E} \cdot \vec{A}}_{\text{Area}}$$



Analogy: Rain accumulating in a barrel.

If 2 inches of rain falls vertically and the opening at the top of a barrel is  $100 \text{ in}^2$ , how much rain falls in the barrel?

$$(2 \text{ inches})(100 \text{ in}^2) = 200 \text{ in}^3$$

What if the rain's path makes a  $30^\circ$  angle with the vertical?

$$(2 \text{ inches})(100 \text{ in}^2)\cos(30^\circ) = 173 \text{ in}^3$$

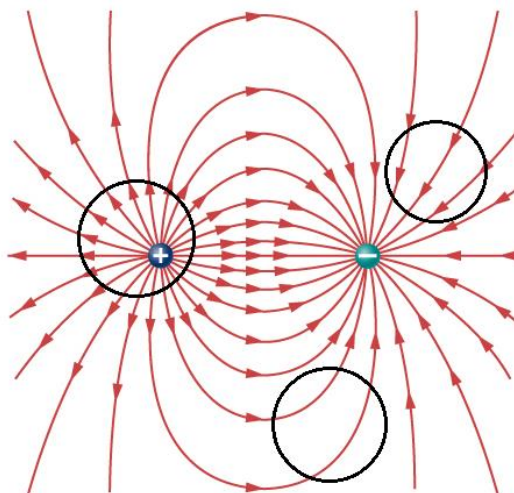
This is a dot product.

"Flux" is very similar in meaning to "flow". What's flowing in  $F_E$ ? Nothing. That's why it's "flux" and not "flow".

### Gauss's Law:

$$\underbrace{\Phi_E}_{\text{Electric Flux}} = \underbrace{\vec{E} \cdot \vec{A}}_{\text{Summed Over Closed Surface}} = \underbrace{\frac{Q_{ENC}}{\epsilon_0}}_{\substack{\text{Enclosed Charge} \\ \text{(divided by a const)}}}$$

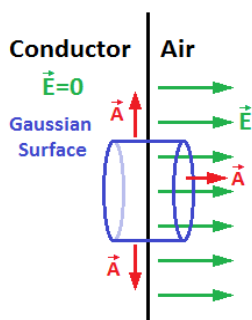
- "Permittivity of Free Space"  $\epsilon_0 = 8.85 \times 10^{-12} \text{ C}^2/(\text{Nm}^2)$   $\epsilon_0 = \frac{1}{4\pi k}$



**Example:** The geometric center of a cube is located at the origin. If a  $30.0 \text{ nC}$  point charge is also at the origin, what is the electric flux that passes through one of the cube's faces?

$$\Phi_E = \frac{1}{6} \cdot \frac{Q_{ENC}}{\epsilon_0} = \frac{30.0 \times 10^{-9} \text{ C}}{(6)[8.85 \times 10^{-12} \text{ C}^2/(\text{Nm}^2)]} = 5.65 \frac{\text{Nm}^2}{\text{C}}$$

**Example:** A conductor has a surface charge density of  $\sigma = 3.25 \mu\text{C}/\text{m}^2$ . Determine the electric field at the surface.



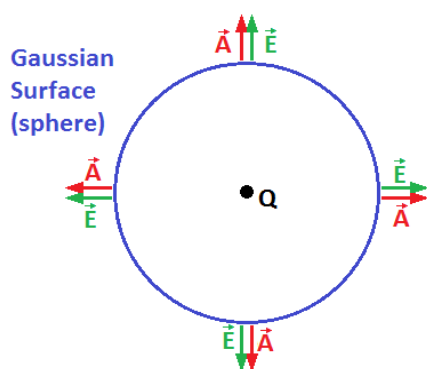
Back Side (circle):  $\vec{E} = 0 \quad \vec{E} \cdot \vec{A} = 0$

Sides (tube):  $\vec{A} \perp \vec{E} \quad \vec{A} \cdot \vec{E} = 0$

Front Side (circle):  $\vec{A} \parallel \vec{E} \quad \vec{E} \cdot \vec{A} = EA$

$$EA = \frac{Q_{ENC}}{\epsilon_0} \quad E = \frac{Q_{ENC}}{\epsilon_0 A} = \frac{\sigma}{\epsilon_0} = \frac{3.25 \times 10^{-6} \text{ C/m}^2}{8.85 \times 10^{-12} \text{ C}^2/(\text{N} \cdot \text{m}^2)} = 367 \text{ kN/C}$$

**Example:** Find the Electric Field due to a point charge Q using Gauss's Law.



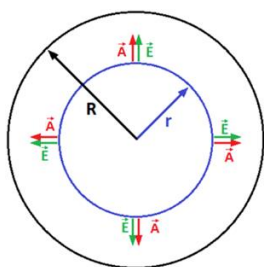
$$\vec{E} \cdot \vec{A} = \frac{Q_{ENC}}{\epsilon_0}$$

$$\vec{A} \parallel \vec{E} \quad \vec{E} \cdot \vec{A} = EA = E(4\pi r^2)$$

$$E(4\pi r^2) = \frac{Q}{\epsilon_0}$$

$$E = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} = k \frac{Q}{r^2}$$

**Example:** A non-conducting sphere of radius R has a uniform charge density,  $\rho$ . Determine the electric field as a function of r.



$$\vec{E} \cdot \vec{A} = \frac{Q_{ENC}}{\epsilon_0}$$

We have spherical symmetry.

$$\vec{A} \parallel \vec{E} \quad \vec{E} \cdot \vec{A} = E \cdot A = E(4\pi r^2)$$

$$\text{For } r < R, \quad Q_{ENC} = \rho \cdot V = \rho \cdot \frac{4}{3}\pi r^3$$

$$E(4\pi r^2) = \frac{Q_{ENC}}{\epsilon_0} \quad E = \frac{Q_{ENC}}{4\pi\epsilon_0 r^2} \quad E = \frac{\rho \cdot \frac{4}{3}\pi r^3}{4\pi\epsilon_0 r^2} \quad E = \frac{\rho r}{3\epsilon_0}$$

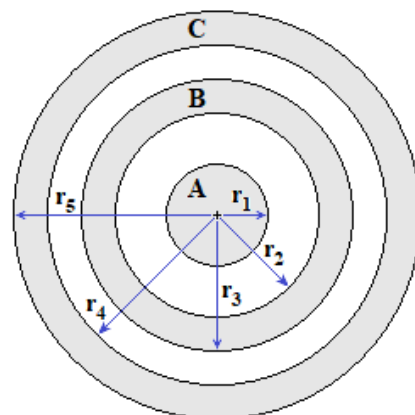
$$\text{For } r > R, \quad Q_{ENC} = \rho \cdot V = \rho \cdot \frac{4}{3}\pi R^3$$

*The only difference between this and our previous  $Q_{ENC}$  is that now our charge doesn't extend out to r, but rather is contained inside the sphere, stopping at R.*

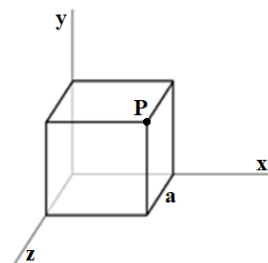
$$E = \frac{Q_{ENC}}{4\pi\epsilon_0 r^2} \quad E = \frac{\rho \cdot \frac{4}{3}\pi R^3}{4\pi\epsilon_0 r^2} \quad E = \frac{\rho R^3}{3\epsilon_0 r^2}$$

**Exercises**

1. A conducting sphere (A) has a radius  $r_1 = 10.0$  cm and a net charge of 22 nC. The sphere (A) is surrounded by two conducting spherical shells. The first shell (B) has an inner radius of  $r_2 = 20.0$  cm, an outer radius of  $r_3 = 25.0$  cm, and a charge of 174 nC. The second shell (C) has an inner radius of  $r_4 = 30.0$  cm, an outer radius of  $r_5 = 35.0$  cm, and a charge of  $-196$  nC. All of these objects are in static equilibrium.

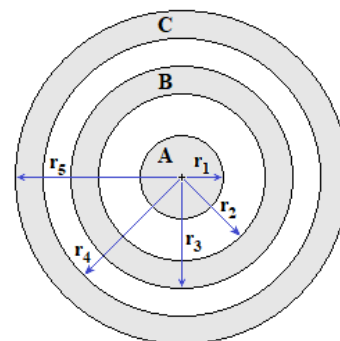


- A. Determine the charge distribution on the surface of the conducting sphere (A).
  - B. Determine the net charge on the inner surface of the first spherical shell (B).
  - C. Determine the net charge on the outer surface of the first spherical shell (B).
  - D. Determine the charge distribution on the outer surface of the second spherical shell (C).
  - E. Determine the magnitude of the electric field at  $r = 28.0$  cm.
  - F. Determine the magnitude of the electric field at  $r = 22.0$  cm.
2. A non-conducting sphere of radius 20.0 cm has 80.0 nC of charge uniformly distributed inside it. Determine the magnitude of the electric field a distance of 10.0 cm from the center of the sphere.
3. A cube with sides of length  $a = 2.00$  m is located in a coordinate system with one vertex at the origin and the opposite vertex at the point P (2.00 m, 2.00 m, 2.00 m). In this region of space there is an electric field given by  $\vec{E} = (\alpha x)\hat{i} + (\beta)\hat{j} + (\gamma z^2)\hat{k}$ , where  $\alpha = 7.00 \frac{N}{C \cdot m}$ ,  $\beta = 5.00 \frac{N}{C}$ , and  $\gamma = 3.00 \frac{N}{C \cdot m^2}$ . Determine the net charge inside the cube.



**Exercise Solutions**

1. A conducting sphere (A) has a radius  $r_1 = 10.0$  cm and a net charge of 22 nC. The sphere (A) is surrounded by two conducting spherical shells. The first shell (B) has an inner radius of  $r_2 = 20.0$  cm, an outer radius of  $r_3 = 25.0$  cm, and a charge of 174 nC. The second shell (C) has an inner radius of  $r_4 = 30.0$  cm, an outer radius of  $r_5 = 35.0$  cm, and a charge of  $-196$  nC. All of these objects are in static equilibrium.



- A. Determine the charge distribution on the surface of the conducting sphere (A).

*As no net charge can stay inside a conductor in static equilibrium, it must be on the surface.*

$$\sigma = \frac{Q}{A} = \frac{Q}{4\pi r_1^2} = \frac{(22 \text{ nC})}{4\pi(0.100 \text{ m})^2} = 175 \frac{\text{nC}}{\text{m}^2}$$

- B. Determine the net charge on the inner surface of the first spherical shell (B).

-22 nC.

*The electric field inside the shell is zero. If we draw a Gaussian sphere just slightly bigger than  $r_2$  then the flux through it will be zero, as  $E=0$  on that surface (inside the conductor). If the flux is zero, the net enclosed charge must also be zero. As there are 22 nC on the inner sphere, there must be -22 nC on the inner surface of conducting shell B.*

- C. Determine the net charge on the outer surface of the first spherical shell (B).

*The total charge on shell B (174 nC) is distributed over the inner and outer surfaces. So the sum of the charges on those two surfaces must be the total charge.*

$$Q_B = Q_{B-In} + Q_{B-Out} \quad Q_{B-Out} = Q_B - Q_{B-In} = 174 \text{ nC} - (-22 \text{ nC}) = 196 \text{ nC}$$

- D. Determine the charge distribution on the outer surface of the second spherical shell (C).

0

*By the same logic as part B, the inner surface of shell C must have the opposite charge as the outer surface of shell B. As the outer surface of shell B holds 196 nC, the inner surface of shell C must have -196 nC. The total charge on shell C (-196 nC) is distributed over the inner and outer surfaces. So the sum of the charges on those two surfaces must be the total charge. As the total charge is found on the inner surface, there is none left for the outer surface.*

- E. Determine the magnitude of the electric field at  $r = 28.0$  cm.

*This radius falls in the gap between shells B and C. Using Gauss's Law on a spherically symmetric object always leads to an electric field that is identical to that the would be created by placing all of the enclosed charge as a point charge at the center.*

$$E = k \frac{Q_{Enc}}{r^2} = \left(9.00 \times 10^9 \frac{N \cdot m^2}{C^2}\right) \frac{(196 \times 10^{-9} C)}{(0.280 m)^2} = 22.5 \frac{kN}{C}$$

F. Determine the magnitude of the electric field at  $r = 22.0$  cm.

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*This is inside shell B, a conductor in static equilibrium. The electric field is always zero inside a conductor in static equilibrium.*

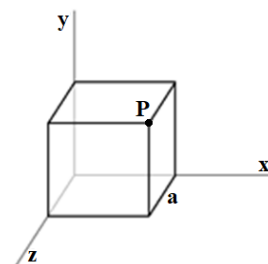
2. A non-conducting sphere of radius 20.0 cm has 80.0 nC of charge uniformly distributed inside it. Determine the magnitude of the electric field a distance of 10.0 cm from the center of the sphere.

*Gauss's Law on a spherically symmetric object always leads to an electric field that is identical to that the would be created by placing all of the enclosed charge as a point charge at the center. As the charge is uniformly distributed, we can find the enclosed charge using the charge density.*

$$Q_{Enc} = \rho V_{Enc} = \left(\frac{Q}{V_{Tot}}\right) (V_{Enc}) = \left(\frac{Q}{\frac{4}{3}\pi R^3}\right) \left(\frac{4}{3}\pi r^3\right) = Q \left(\frac{r}{R}\right)^3 = (80.0 nC) \left(\frac{10.0 cm}{20.0 cm}\right)^3 = 10.0 nC$$

$$E = k \frac{Q_{Enc}}{r^2} = \left(9.00 \times 10^9 \frac{N \cdot m^2}{C^2}\right) \frac{(10 \times 10^{-9} C)}{(0.100 m)^2} = 9.00 \frac{kN}{C}$$

3. A cube with sides of length  $a = 2.00$  m is located in a coordinate system with one vertex at the origin and the opposite vertex at the point P (2.00 m, 2.00 m, 2.00 m). In this region of space there is an electric field given by  $\vec{E} = (\alpha x)\hat{i} + (\beta)\hat{j} + (\gamma z^2)\hat{k}$ , where  $\alpha = 7.00 \frac{N}{C \cdot m}$ ,  $\beta = 5.00 \frac{N}{C}$ , and  $\gamma = 3.00 \frac{N}{C \cdot m^2}$ . Determine the net charge inside the cube.



$$\sum \vec{E} \cdot \vec{A} = \frac{Q_{Enc}}{\epsilon_0} \quad Q_{Enc} = \epsilon_0 \sum \vec{E} \cdot \vec{A}$$

*Using Gauss's Law, we must calculate the flux through each of the 6 faces of the cube and then sum those.*

Right Face:  $\vec{A} = A\hat{i}$   $\Phi_1 = \vec{E} \cdot \vec{A} = [(\alpha x)\hat{i} + (\beta)\hat{j} + (\gamma z^2)\hat{k}] \cdot A\hat{i} = \alpha x \cdot A = \alpha a^3$   
 $\Phi_1 = \alpha a^3$ . *x is constant over the surface (x=a), and A is just the area of the face (a<sup>2</sup>).*

Left Face:  $\vec{A} = -A\hat{i}$   $\Phi_2 = \vec{E} \cdot \vec{A} = [(\alpha x)\hat{i} + (\beta)\hat{j} + (\gamma z^2)\hat{k}] \cdot -A\hat{i} = -\alpha x \cdot A = 0$   
 $\Phi_2 = 0$ . *As x=0 on the surface, the electric flux (Φ<sub>2</sub>) is zero.*

Front Face:  $\vec{A} = A\hat{k}$   $\Phi_3 = \vec{E} \cdot \vec{A} = [(\alpha x)\hat{i} + (\beta)\hat{j} + (\gamma z^2)\hat{k}] \cdot A\hat{k} = \gamma z^2 A = \gamma a^4$   
 $\Phi_3 = \gamma a^4$ . *z is constant over the surface (z=a), and A is just the area of the face (a<sup>2</sup>).*

Back Face:  $\vec{A} = -A\hat{k}$   $\Phi_4 = \vec{E} \cdot \vec{A} = [(\alpha x)\hat{i} + (\beta)\hat{j} + (\gamma z^2)\hat{k}] \cdot -A\hat{k} = -\gamma z^2 A = 0$   
 $\Phi_4 = 0$ . *As z=0 on the surface, the electric flux (Φ<sub>2</sub>) is zero.*

$$\text{Top Face: } \vec{A} = A\hat{j} \quad \Phi_5 = \vec{E} \cdot \vec{A} = [(\alpha x)\hat{i} + (\beta)\hat{j} + (\gamma z^2)\hat{k}] \cdot A\hat{j} = \beta A = \beta a^2$$

$$\Phi_5 = \beta a^2. \quad A \text{ is just the area of the face } (a^2).$$

$$\text{Bottom Face: } \vec{A} = -A\hat{j} \quad \Phi_6 = \vec{E} \cdot \vec{A} = [(\alpha x)\hat{i} + (\beta)\hat{j} + (\gamma z^2)\hat{k}] \cdot -A\hat{j} = -\beta A = -\beta a^2$$

$$\Phi_6 = -\beta a^2. \quad A \text{ is just the area of the face } (a^2).$$

$$Q_{Enc} = \epsilon_0 \sum \vec{E} \cdot \vec{A} = \epsilon_0 (\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5 + \Phi_6)$$

$$Q_{Enc} = \epsilon_0 (\alpha a^3 + 0 + \gamma a^4 + 0 + \beta a^2 - \beta a^2) = \epsilon_0 (\alpha a^3 + \gamma a^4)$$

$$Q_{Enc} = \left( 8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2} \right) \left[ \left( 7.00 \frac{N}{C \cdot m} \right) (2.00 \text{ m})^3 + \left( 3.00 \frac{N}{C \cdot m^2} \right) (2.00 \text{ m})^4 \right]$$

$$Q_{Enc} = 920.4 \text{ pC}$$